Fluid Mechanics
Unit 3- Flow Kinematics

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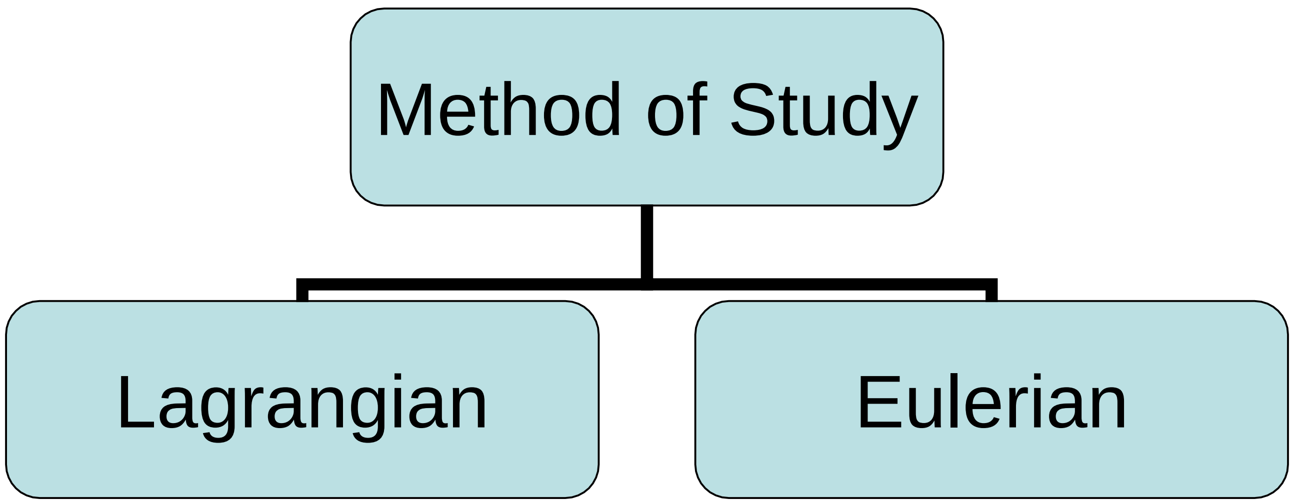
Fluid Kinematics

Study of fluid motion without consideration of forces



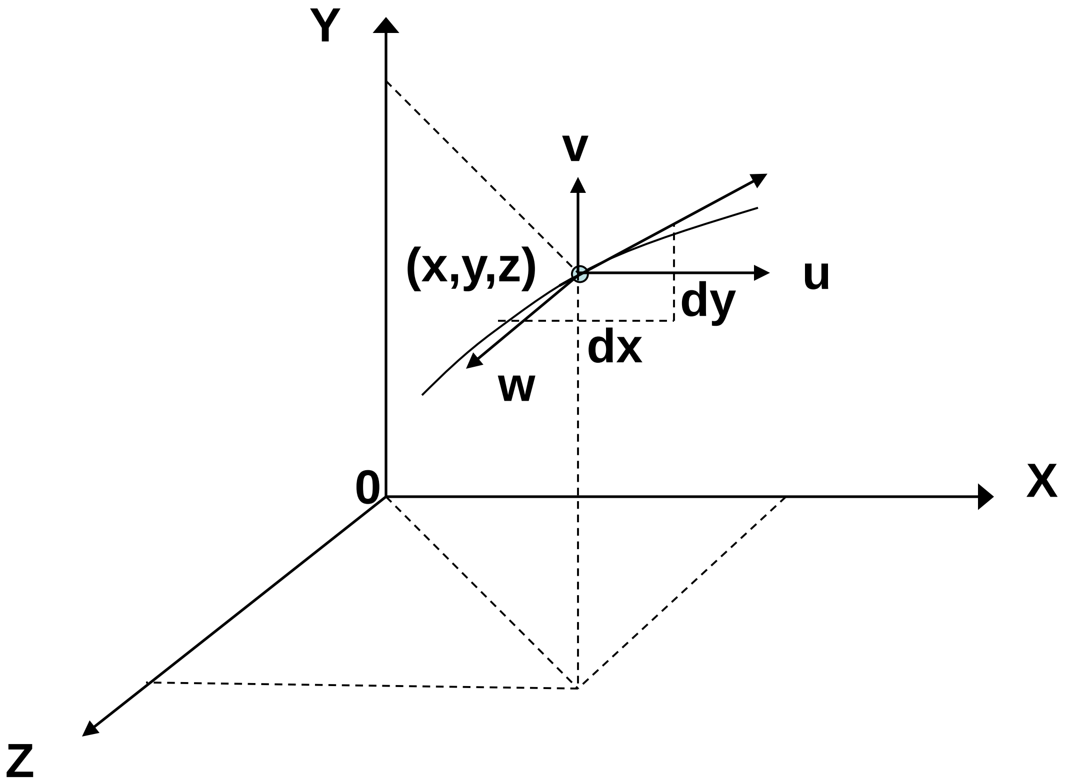
Lagrangian – individual fluid particle is selected

Eulerian – any point occupied by fluid is selected

Velocity of Fluid Particles

Solids – The body as whole can be considered for determination of velocity

Fluids – Motion of fluids can be different at different points



$$V = \lim_{dt \to 0} \frac{ds}{dt} \;;$$

$$u = \lim_{dt \to 0} \frac{dx}{dt} \;,$$

$$v = \lim_{dt \to 0} \frac{dy}{dt} \;,$$

$$w = \lim_{dt \to 0} \frac{dz}{dt}$$

Velocity of Fluid Particles

$$V = f(x, y, z, t)$$

$$u = f(x, y, z, t)$$

$$v = f(x, y, z, t)$$

$$w = f(x, y, z, t),$$

Vector notation ⇨ $V = iu + jv + kw$

i, j **and** k **are the unit vectors parallel to X, Y and Z axis**

Types of Fluid Flow

- Steady and unsteady flow
- Uniform and Non-uniform flow
- One dimensional, two-dimensional and three dimensional flow
- Rotational and Irrotational flow
- Laminar and Turbulent flow

Types of Fluid Flow

Steady flow – if at any point in the flow system of a fluid the various characteristics such as velocity, pressure, density, temperature etc. which describe the behavior of the fluid flow, do not change with time then the flow is called steady flow

Mathematically, $\frac{dV}{dt}=0, \ \frac{du}{dt}=0, \ \frac{dv}{dt}=0, \ \frac{dw}{dt}=0, \ \frac{dp}{dt}=0, \ \frac{d\rho}{dt}=0,$

Unsteady flow - – if at any point in the flow system of a fluid the various characteristics such as velocity, pressure, density, temperature etc. which describe the behavior of the fluid flow, change with time then the flow is called steady flow

Mathematically, $\frac{dV}{dt}\neq 0, \ \frac{du}{dt}\neq 0, \ \frac{dv}{dt}\neq 0, \ \frac{dw}{dt}\neq 0, \ \frac{dp}{dt}\neq 0, \ \frac{d\rho}{dt}\neq 0,$

Types of Fluid Flow

Uniform flow – If there is **no change** in the magnitude and direction of velocity **with respect to space** in the flow system i. e velocity does not change from point to point then the flow is called uniform flow

Mathematically, $$\frac{dV}{ds}=0,\quad \frac{du}{ds}=0,\quad \frac{dv}{ds}=0,\quad \frac{dw}{ds}=0$$

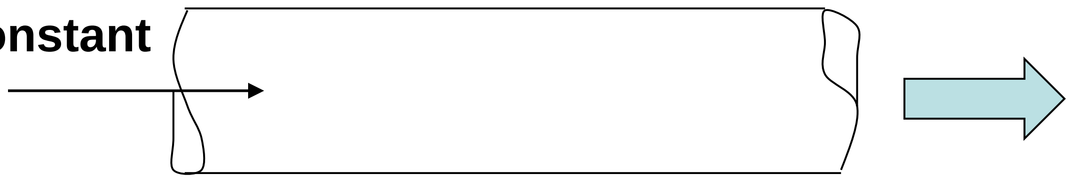
Non-uniform flow - If the magnitude and direction of velocity **changes with respect to space** in the flow system i. e velocity from point to point is different then the flow is called non-uniform flow

Mathematically, $$\frac{dV}{ds}\neq 0,\quad \frac{du}{ds}\neq 0,\quad \frac{dv}{ds}\neq 0,\quad \frac{dw}{ds}\neq 0$$

Types of Fluid Flow

Combination of steady, unsteady, uniform and non-uniform flow is possible as these can exist independently

Q = Constant



Steady uniform flow

Flow through a long pipe of constant diameter at constant rate

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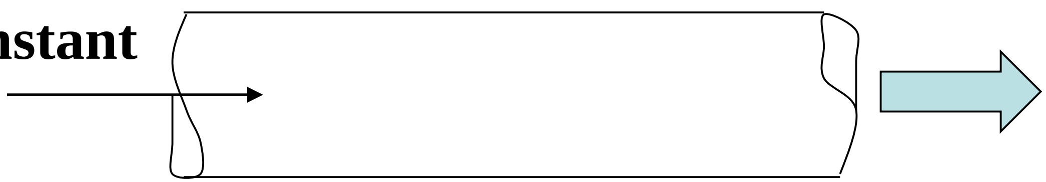
$Q \neq$ **Constant**



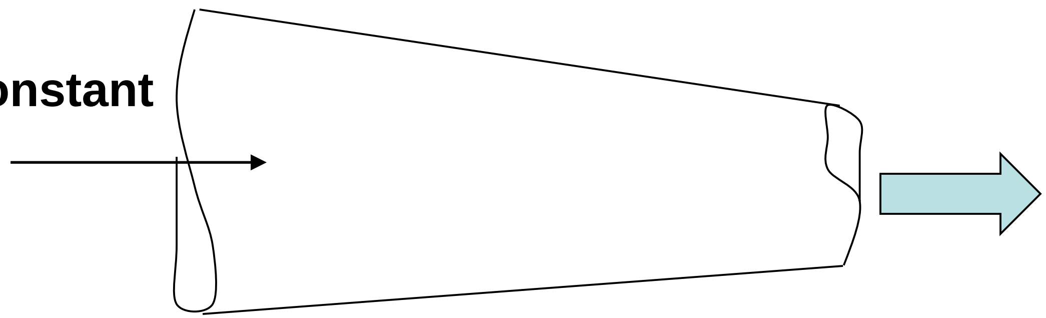
Unsteady uniform flow

Flow through a long pipe of constant diameter at varying rate

Types of Fluid Flow

Combination of steady, unsteady, uniform and non-uniform flow is possible as these can exist independently

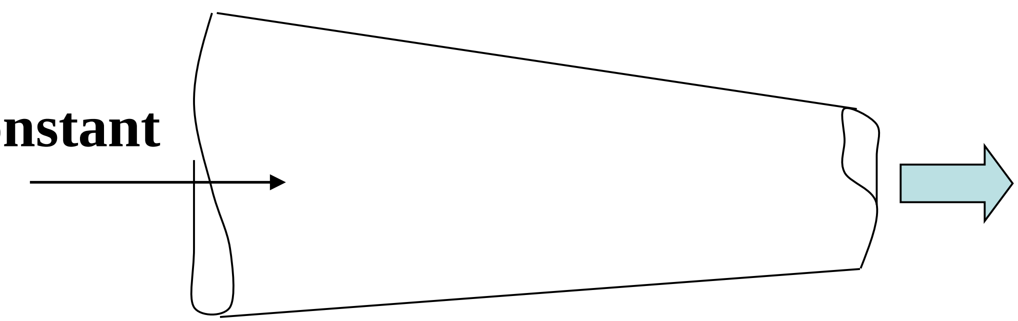
Q = Constant



Steady non-uniform flow

Flow through a tapered pipe at constant rate

$Q \neq$ **Constant**

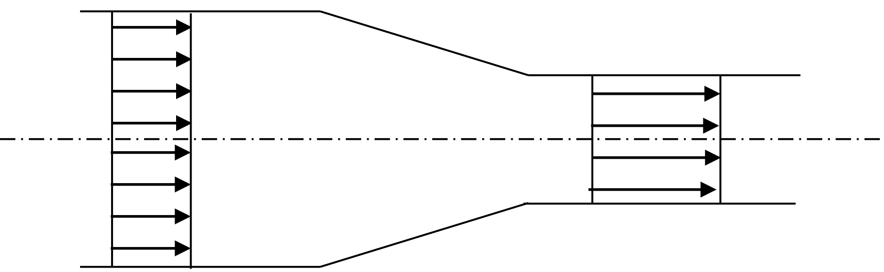


Unsteady non-uniform flow

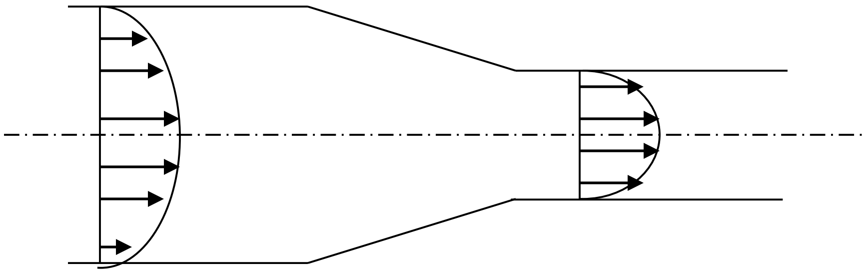
Flow through a tapered pipe at varying rate

Types of Fluid Flow

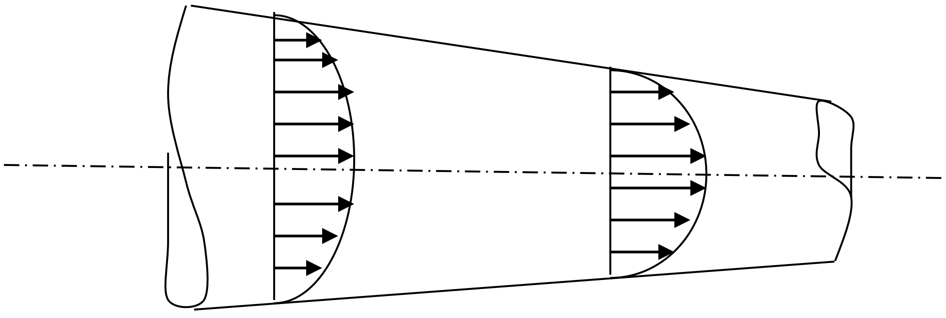
	Three-dimensional	Two-dimensional	One-dimensional
Steady flow	$V = f(x, y, z)$	$V = f(x, y)$	$V = f(x)$
unsteady flow	$V = f(x, y, z, t)$	$V = f(x, y, t)$	$V = f(x, t)$



One dimensional flow



Two dimensional flow



Three dimensional flow

Types of Fluid Flow

- **Rotational flow – A flow is said to be rotational if the fluid particles while moving in the flow direction rotate about their mass centre.**

- **Irrotational flow – A flow is said to be rotational if the fluid particles while moving in the flow direction do not rotate about their mass centre.**
- **The true irrotational flow exits only in the case of ideal fluid for which no tangential or shear stress occur**
- **The flow of low viscosity fluid may be considered as irrotational flow**

Types of Fluid Flow

- **Laminar flow – The flow is said to be laminar when the various fluid particles move in layers (or laminae) with one layer of fluid sliding smoothly over an adjacent layer**

- **Viscosity plays a significant role**

- **Flow of a viscous fluid may be treated as laminar flow**

- **Turbulent flow – The fluid particles move in an entirely haphazard or disorderly manner that results in rapid and continuous mixing of fluid leading to momentum transfer as the flow occurs**

- **Eddies of vortices of different sizes and shapes are present which move over large distances.**

- **These eddies causes fluctuations in velocities and pressures which are functions of time**

Types of Fluid Flow

Are all turbulent flows unsteady?

If temporal mean values of velocities and pressure considered over a large time span are constant then the turbulent flow can also be considered as a steady flow

The occurrence of turbulent flow is more frequent that laminar flow

Flow in natural streams, artificial channels, water supply pipes, sewers etc are the best examples of turbulent flow

Flow Pattern Description

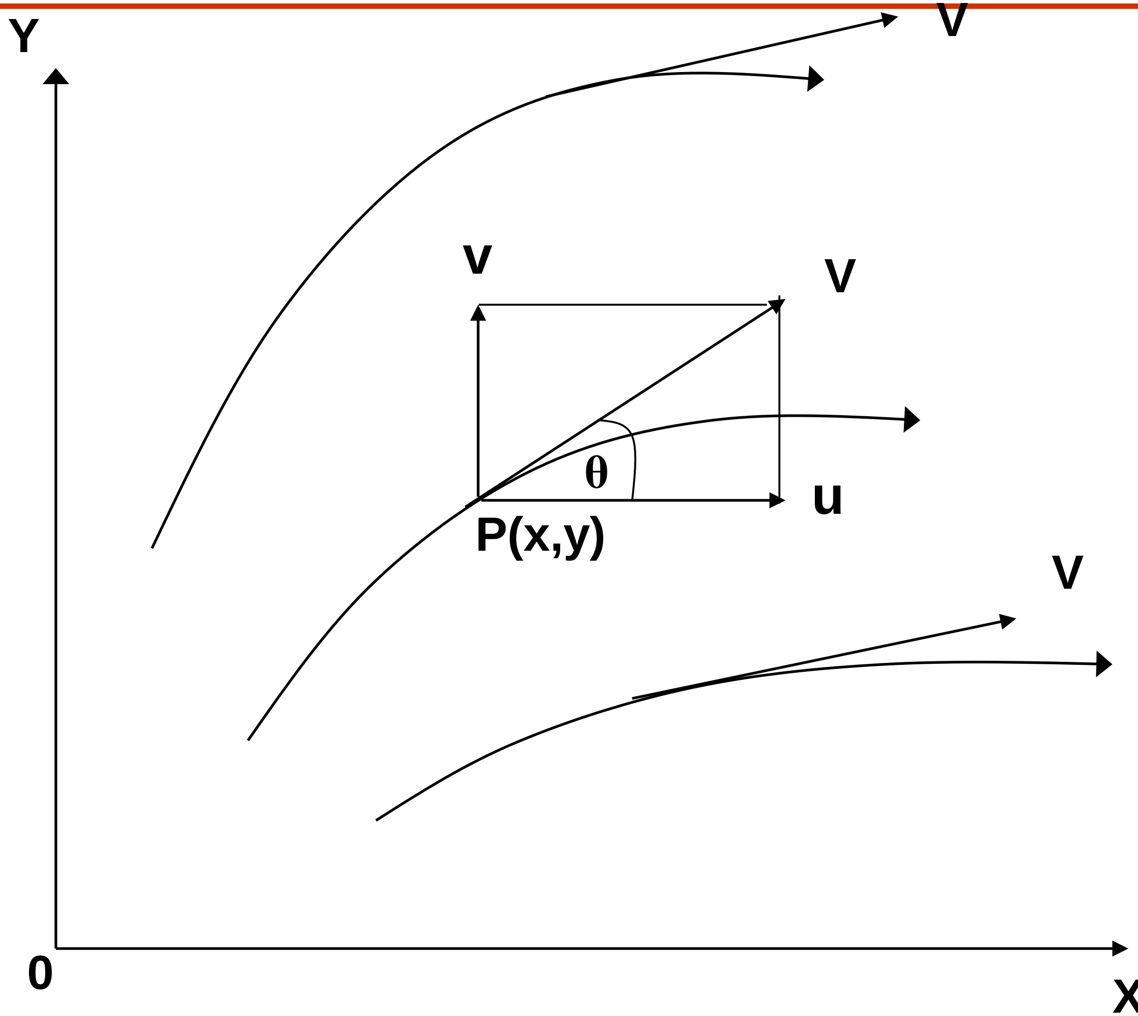
- The flow pattern description helps to understand the physical aspects of flow
- It helps to understand the effect of governing parameters such as geometry of flow system and flow parameters (velocity of flow, density of fluid, viscosity of fluid etc.) on the flow physics

The flow pattern may be described by

- Streamlines
- Stream tubes
- Streak-lines
- Path lines
- Vorticity contours
- Wake

The vorticity contours and wake structures are being recently used by many researchers for exploring the physical phenomenon [Sewatkar et al. 2011 (Physics of Fluids) and Sewatkar et al. 2011 (journal of Fluid Mechanics)

Streamlines



Streamline is an imaginary curve drawn through a flowing fluid such that a tangent to it at any point gives the direction of the velocity of flow at that point.

Streamlines

Mathematical representation of streamline

$$\frac{v}{u} = \tan\theta = \frac{dy}{dx}$$

dy and dx are the y and x components of the differential displacement along the streamline in the vicinity of P

$$\frac{dx}{u} = \frac{dy}{v}; \ \textbf{or} \ (udy - vdx) = 0$$

For three dimensional flow

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

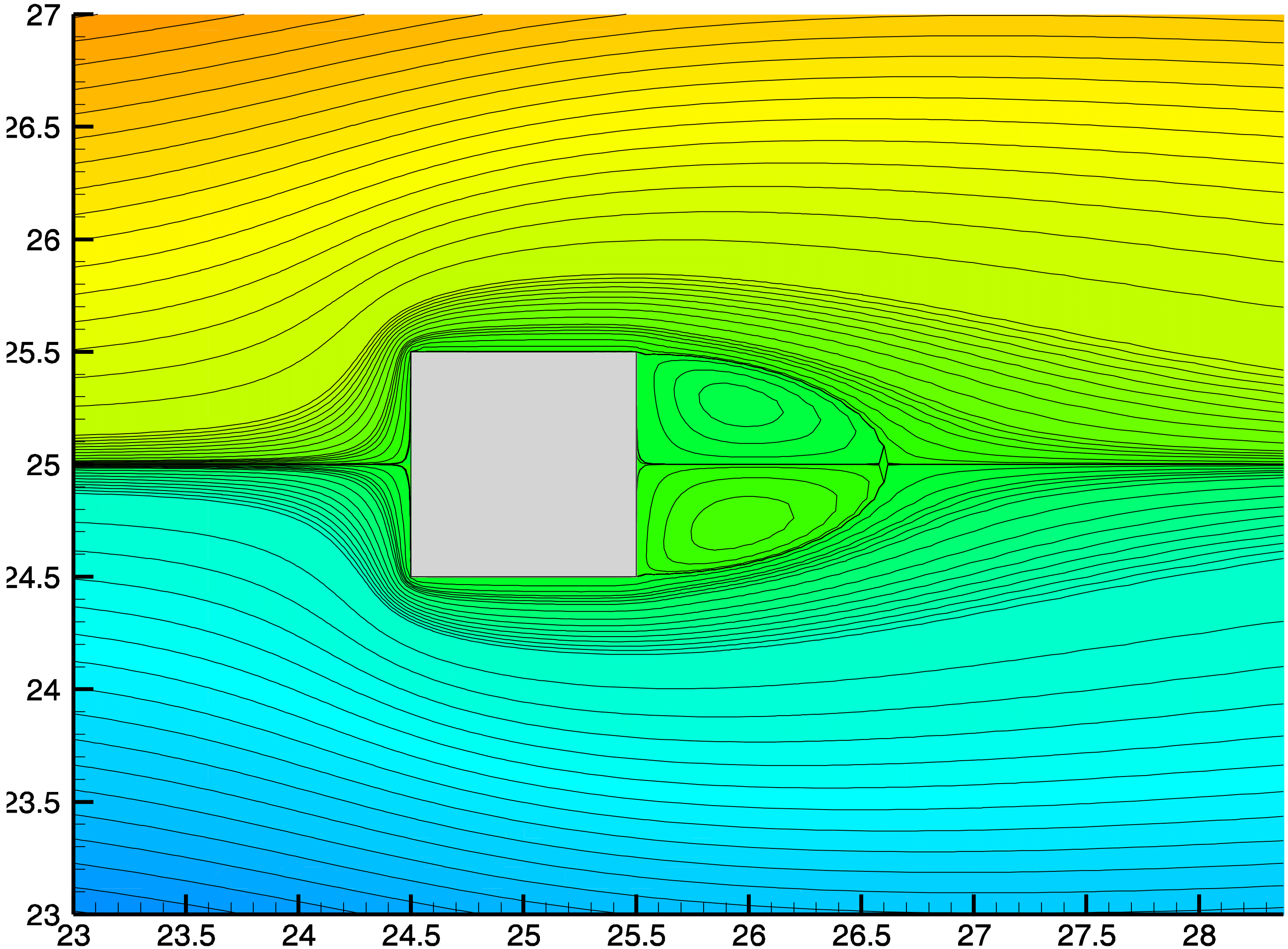
Characteristics Streamline

There can be no component of the velocity at right angles to the streamline and hence there can be no flow across any streamline

In steady flow since there is no change in the direction of the velocity vector at any point, the flow pattern is not changing

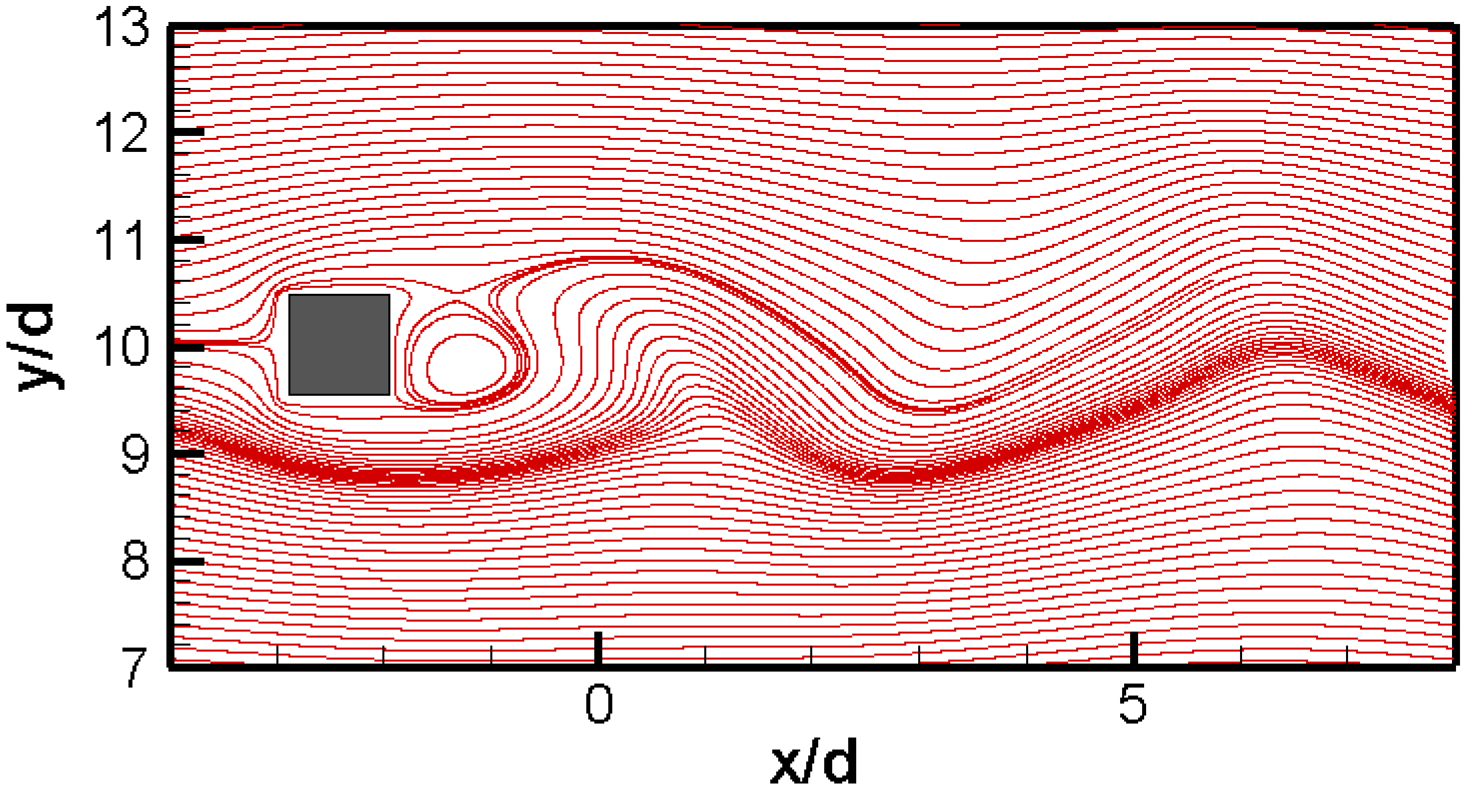
In unsteady flow since there is continuous change in the direction of the velocity vector at all points, the flow pattern is continuously alters. Hence, the streamline pattern in such a case is called instantaneous streamline pattern

Streamline Pattern for single Square Cylinder



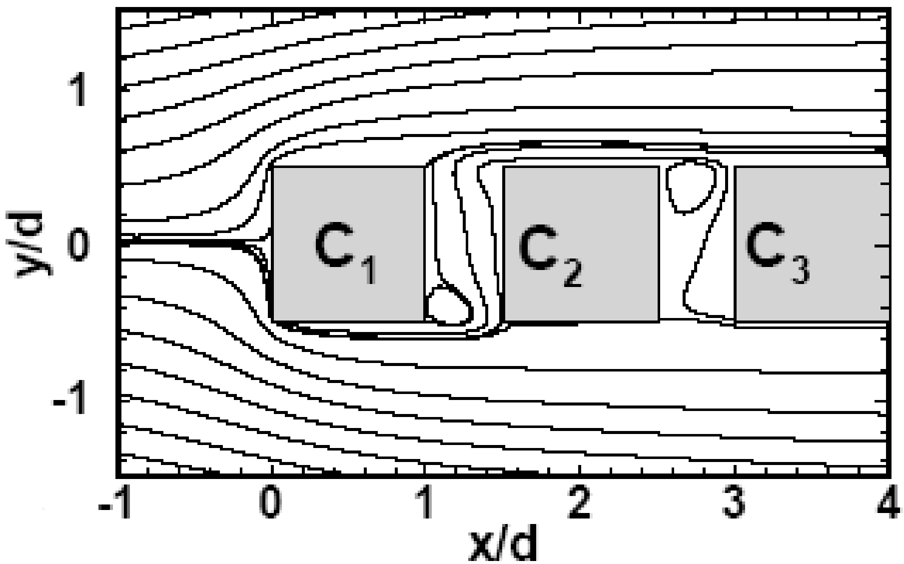
Steady flow around square cylinder at Re = 20

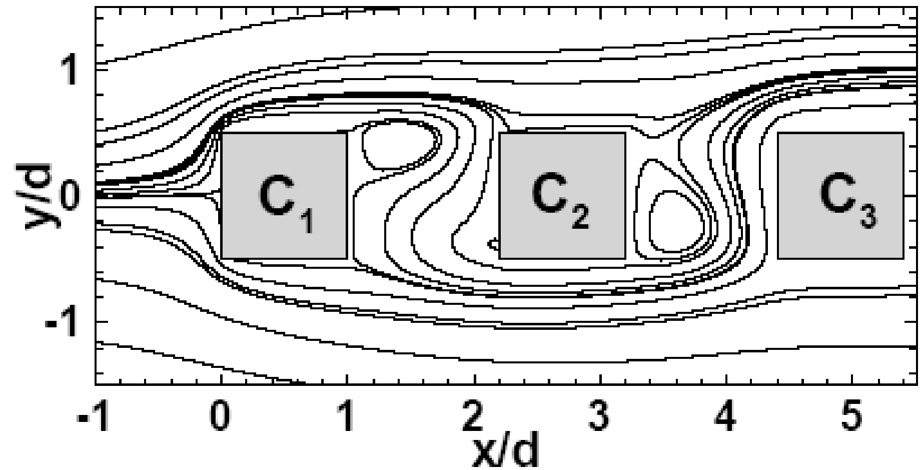
Streamline Pattern for single Square Cylinder

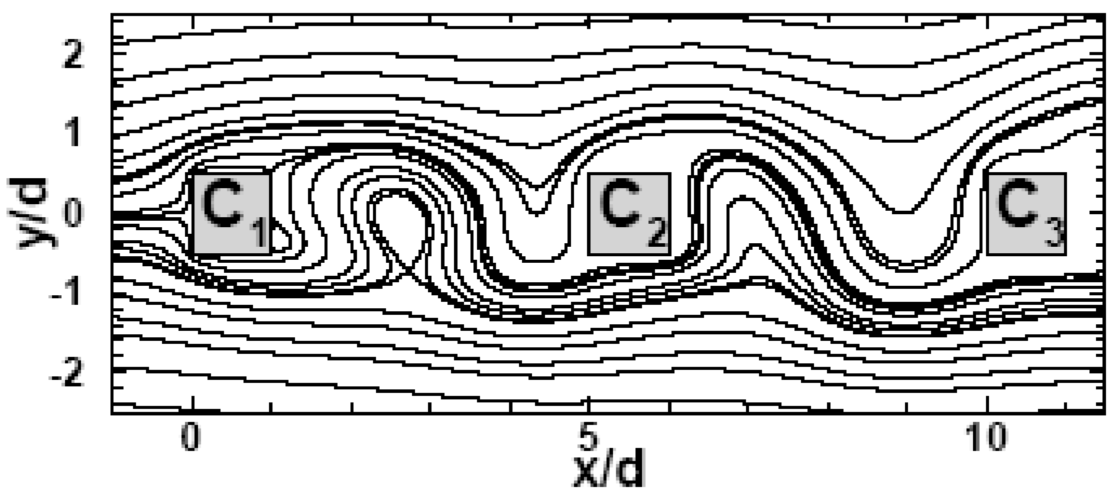


Steady flow around square cylinder at Re = 80

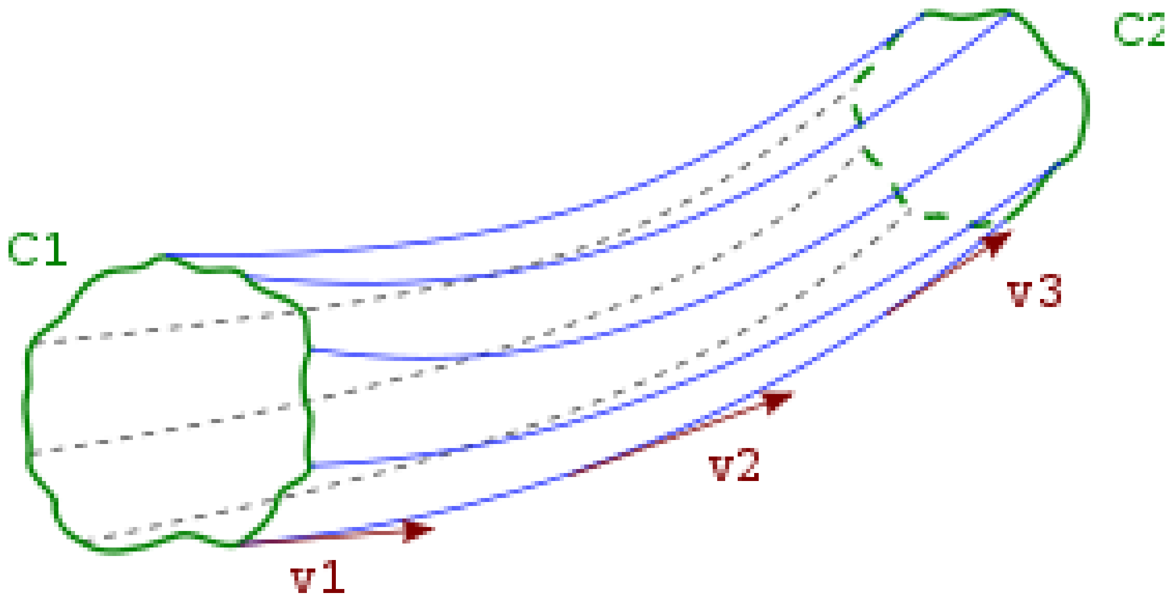
Streamlines Pattern for In-line Square Cylinders







Stream Tube



- A stream tube is a tube imagined to be formed by a group of streamlines passing through a small closed curve, which may or may not be circular
- All the characteristics of stream lines are equally applicable to stream tube
- There can be flow in or out at the ends but flow can not be across the tube

Path line

Path line is the line traced by a single fluid particle as it moves over a period of time

Path line provides direction of a same fluid particle at different successive instances, while streamline provides direction of different fluid particles at same instant

For steady flow path-lines and streamlines are identical

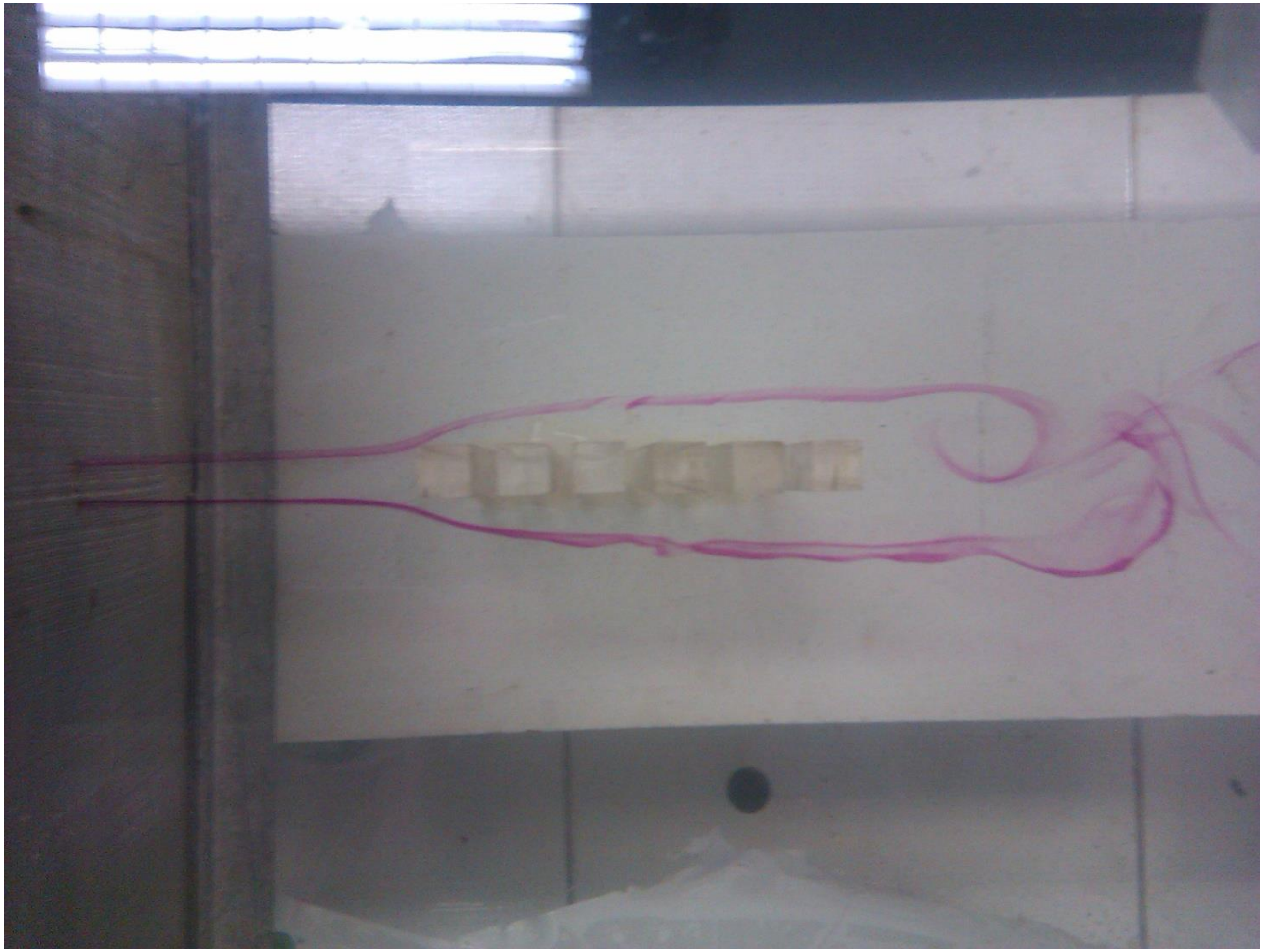
Streak line

Streak line is the line traced by the fluid particles passing through a fixed point in the flow system

Please refer to the figure discussed on the black board in the class

For steady flow streak-lines, path-lines and streamlines are identical

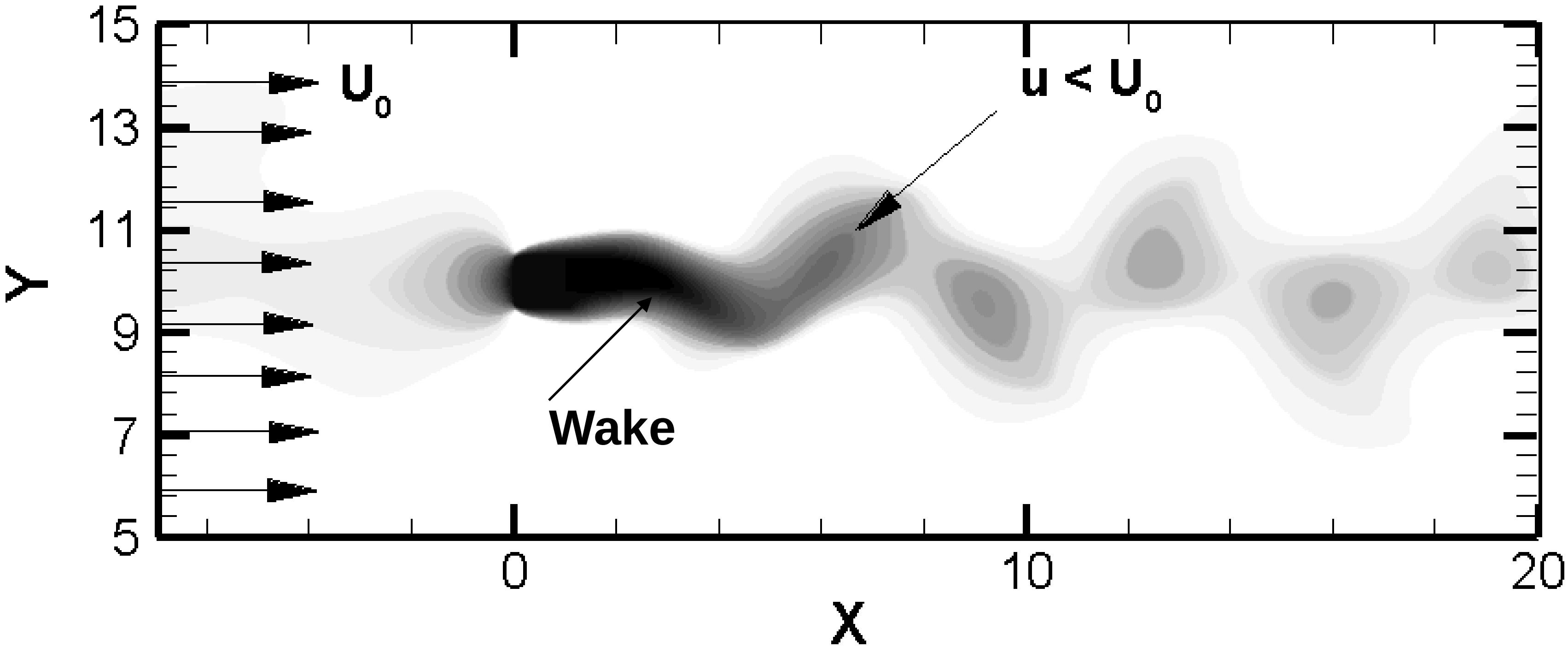
Streak line



Dye visualization for flow around in-line square cylinders at gap ratio = 0.5 and Re = 100

Wake

Wake is the region in the flow at which the streamwise velocity is less than free stream velocity (Zdravkovich, 1997)

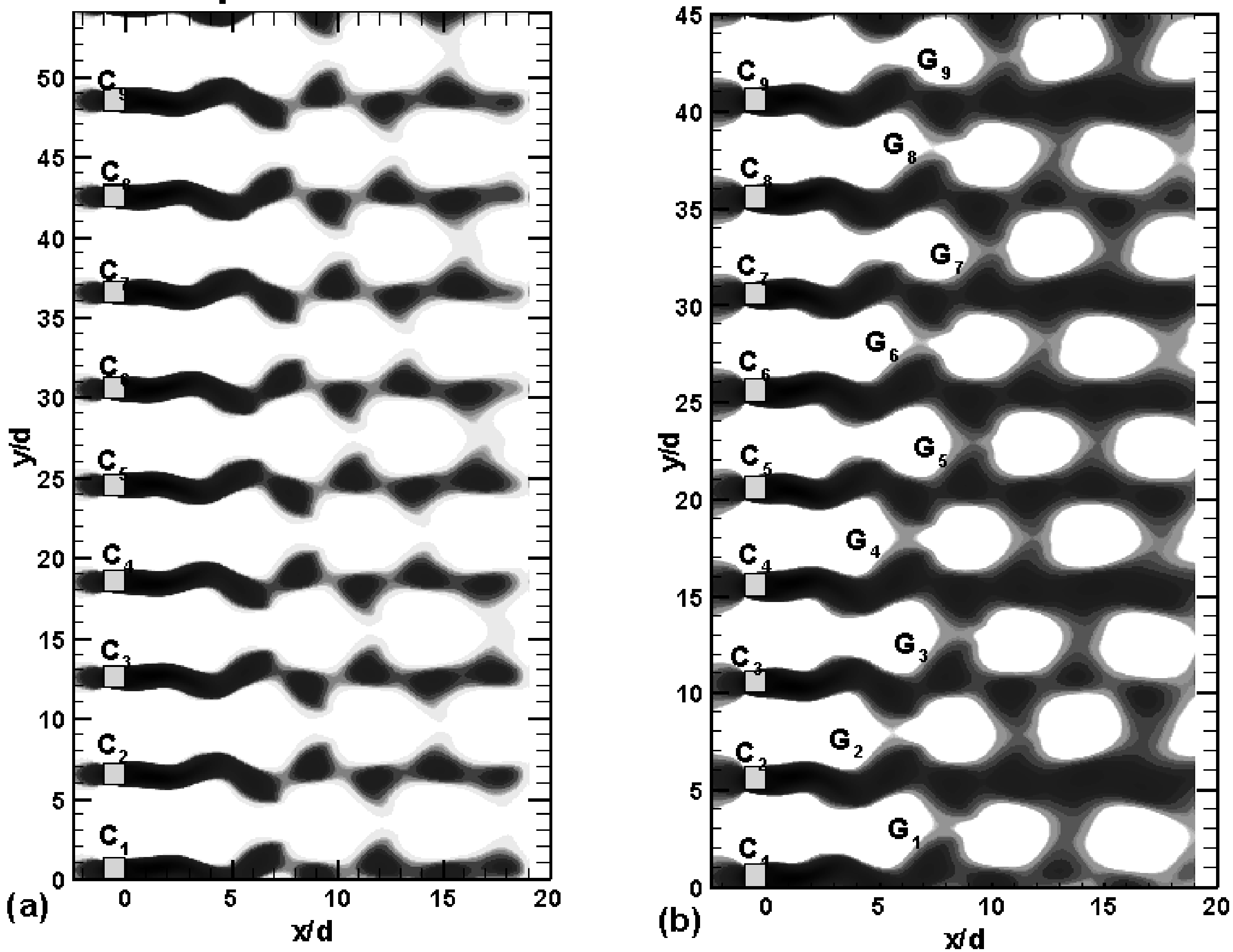


Flow across single cylinder at Re = 80

Wake

Gap ratio = 5

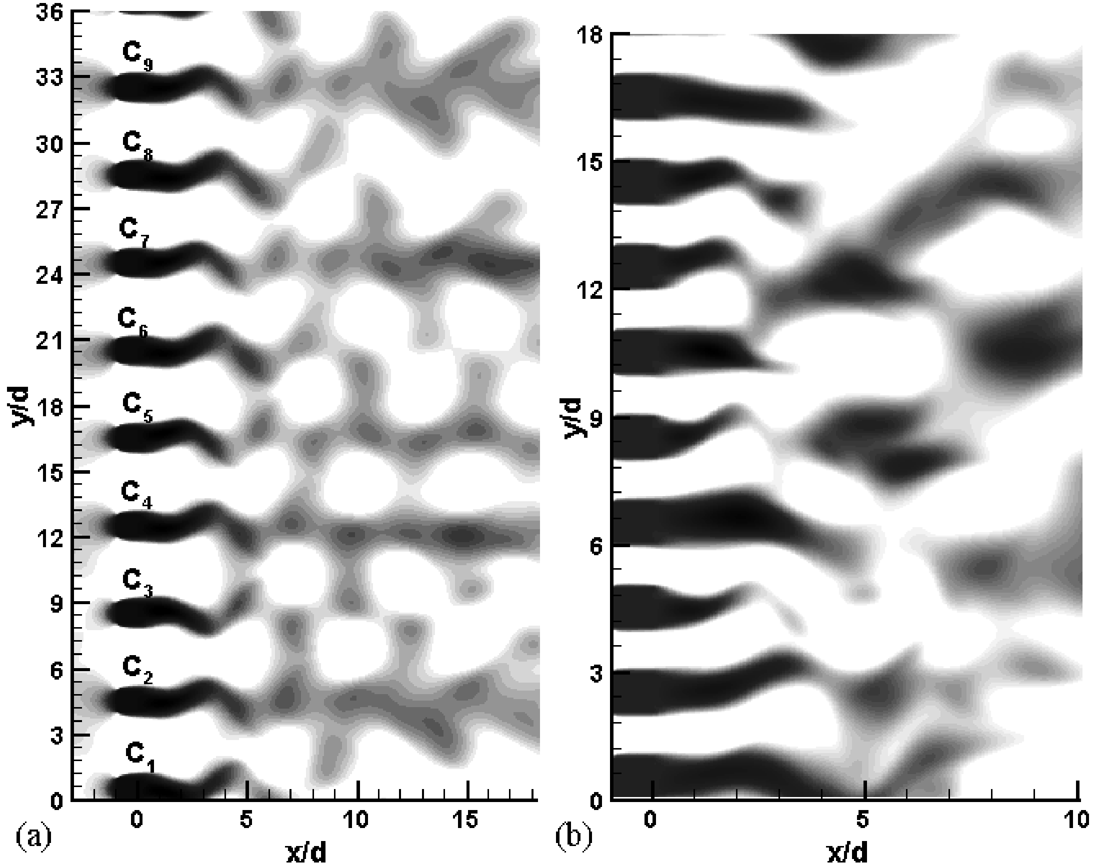
Gap ratio = 4



Flow across large number of cylinders at Re = 80, (Sewatkar et al. 2009, Physics of Fluids)

Wake

Gap ratio = 3 Gap ratio = 1



Flow across large number of cylinders at Re = 80 (Sewatkar et al. 2009, Physics of Fluids)

Basic Principles of Fluid Flow

Principle of Conservation of Mass → **Continuity Equation**

Principle of Conservation of Momentum → **Momentum Equation**

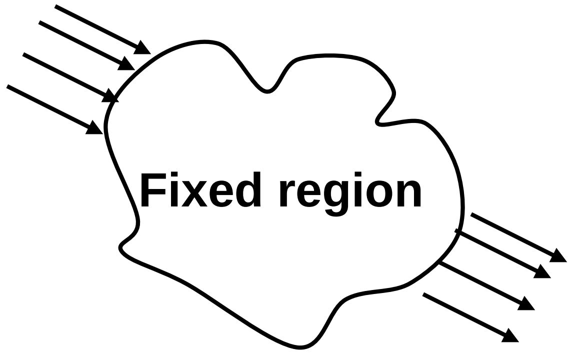
Fluid Mechanics

Principle of Conservation of Energy → **Energy Equation**

Heat Transfer

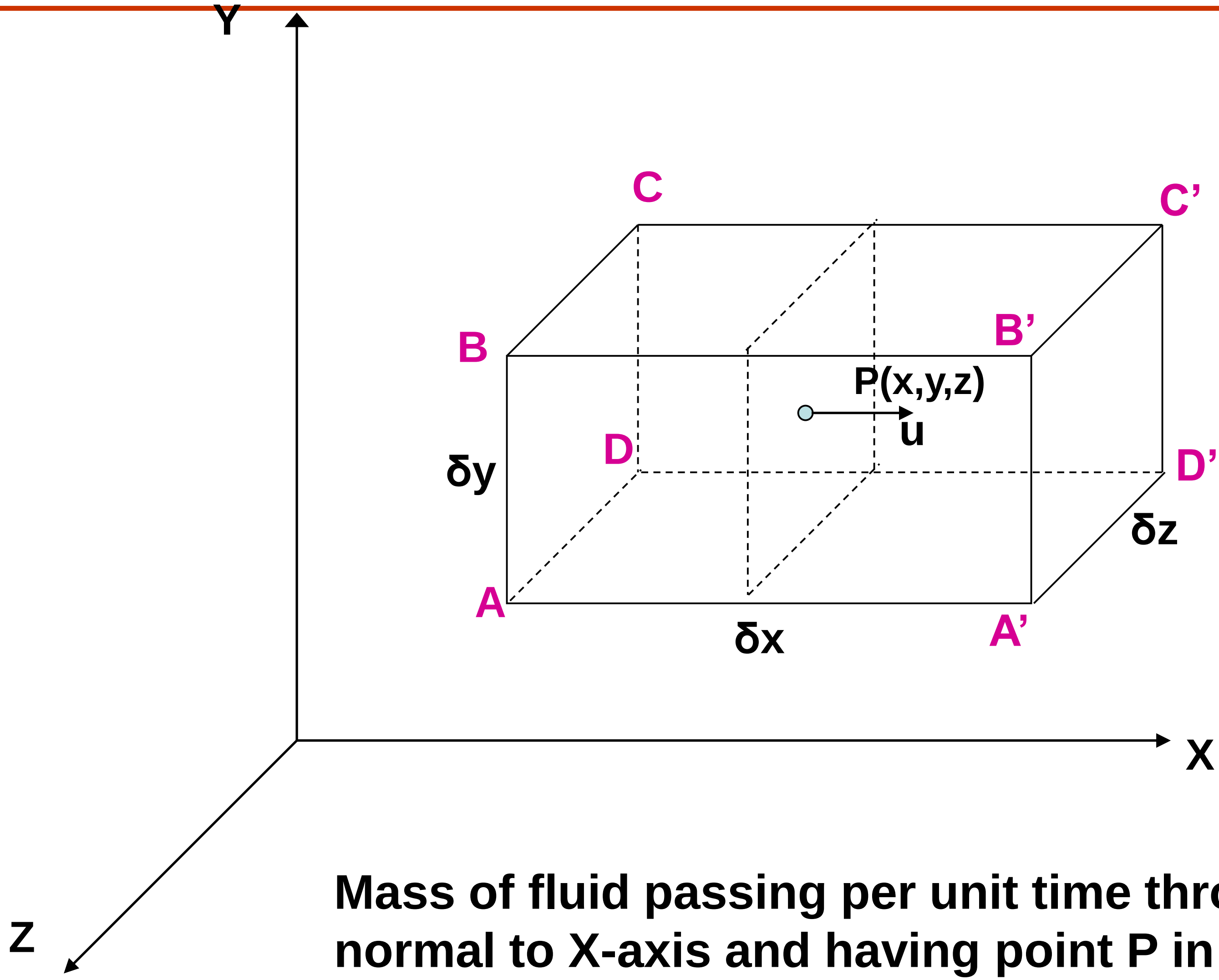
Continuity Equation

Continuity equation is the mathematical expression for law of conservation of mass



$$\begin{bmatrix}\text{Rate of increase or} \\ \text{decrease of fluid mass} \\ \text{within fixed region}\end{bmatrix} = \begin{bmatrix}\text{Rate of mass flow} \\ \text{at the entrance}\end{bmatrix} \pm \begin{bmatrix}\text{Rate of mass flow} \\ \text{at the exit}\end{bmatrix}$$

Continuity Equation in Cartesian Coordinates



ρ = density of fluid

Mass of fluid passing per unit time through the face normal to X-axis and having point P in it

$$= (\rho u \delta y \delta z)$$

Continuity Equation in Cartesian Coordinates

Mass of fluid passing per unit time through the face ABCD in X-direction

$$=\left[(\rho u\delta y\delta z)+\frac{\partial}{\partial x}(\rho u\delta y\delta z)\left(-\frac{\delta x}{2}\right)\right]$$

Mass of fluid passing per unit time through the face A'B'C'D' in X-direction

$$=\left[(\rho u\delta y\delta z)+\frac{\partial}{\partial x}(\rho u\delta y\delta z)\left(\frac{\delta x}{2}\right)\right]$$

Net mass of fluid that has remained in the parallelepiped per unit time along X-direction

$$=\left[(\rho u\delta y\delta z)-\frac{\partial}{\partial x}(\rho u\delta y\delta z)\left(\frac{\delta x}{2}\right)\right]-\left[(\rho u\delta y\delta z)+\frac{\partial}{\partial x}(\rho u\delta y\delta z)\left(\frac{\delta x}{2}\right)\right]$$

$$=-\frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z$$

Continuity Equation in Cartesian Coordinates

Similarly

Net mass of fluid that has remained in the parallelepiped per unit time along Y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\delta x \delta y \delta z$$

Net mass of fluid that has remained in the parallelepiped per unit time along Z-direction

$$= -\frac{\partial}{\partial z}(\rho w)\delta x \delta y \delta z$$

Net total mass of fluid that has remained in the parallelepiped per unit time

$$= -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x \delta y \delta z$$

Equation A

Continuity Equation in Cartesian Coordinates

Mass of fluid in the parallelepiped at any instant

$$= \rho(\delta x \delta y \delta z)$$

Rate of increase of mass of fluid in the parallelepiped with time

$$= \frac{\partial}{\partial t}\rho(\delta x \delta y \delta z) = \frac{\partial \rho}{\partial t}(\delta x \delta y \delta z)$$ **Equation B**

Equate Equation A and B

$$\frac{\partial \rho}{\partial t}(\delta x \delta y \delta z) = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x \delta y \delta z$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

The most general form of continuity equation applicable to all types of flows

Continuity Equation in Cartesian Coordinates

For steady flow

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

For incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

In vector notation the generalized equation is written as

$$\nabla \cdot (\rho V) = 0$$

Home work : Obtain the generalized continuity equation for spherical and cylindrical coordinates

Continuity Equation in Cartesian Coordinates

Continuity equation for two dimensional flow

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

Generalized 2D continuity equation

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

2D continuity equation for steady flow

$$\frac{\partial(u)}{\partial x}+\frac{\partial(v)}{\partial y}=0$$

2D continuity equation for steady flow of an incompressible fluid

Continuity Equation in Cartesian Coordinates

Continuity equation for one dimensional flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

Generalized 1D continuity equation

$$\frac{\partial (\rho u)}{\partial x} = 0$$

1D continuity equation for steady flow

$$\frac{du}{dx} = 0$$

1D continuity equation for steady flow of an incompressible fluid

Continuity Equation for 1D flow

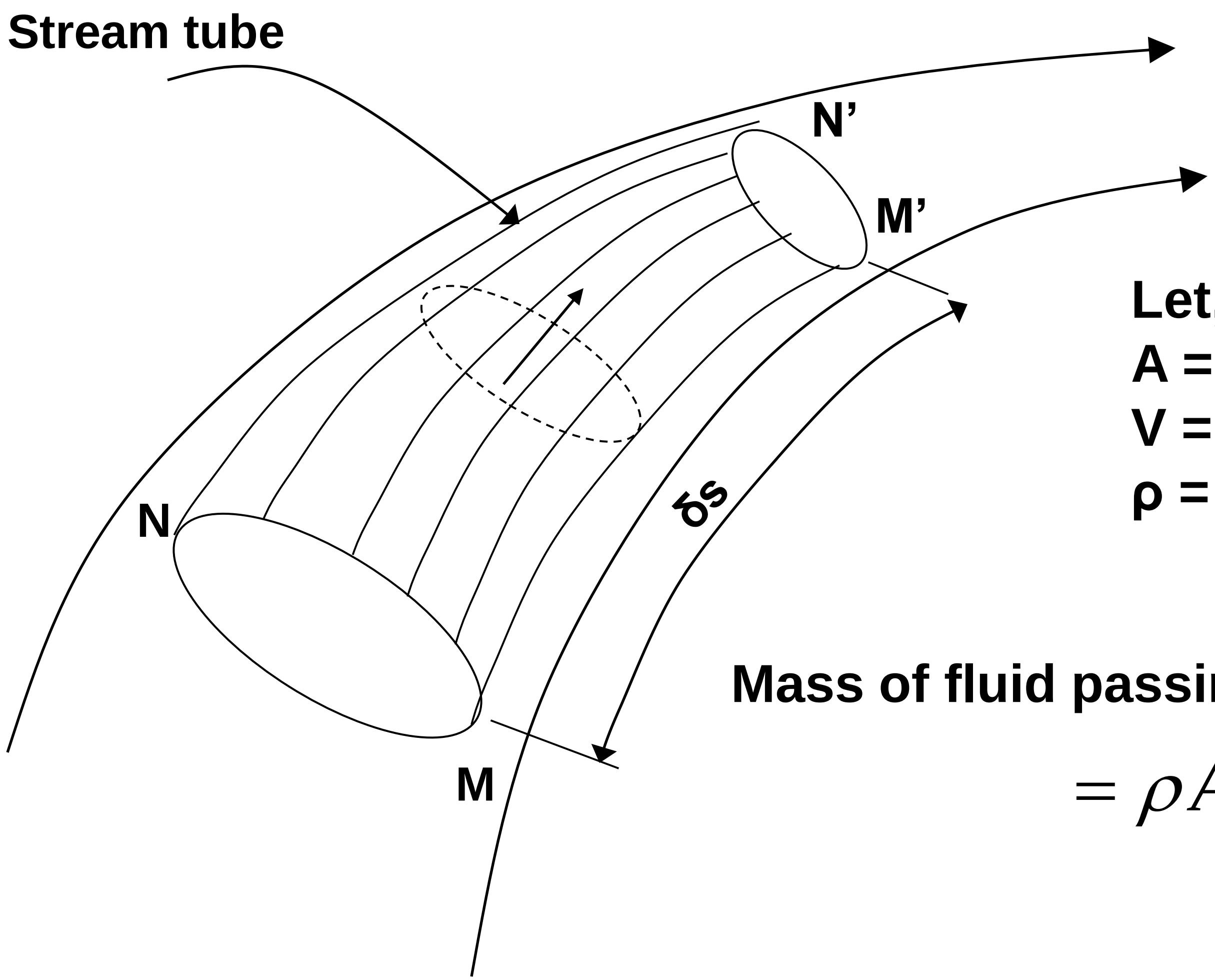
- **One-dimensional Steady flow continuity equation** $$\frac{\partial(\rho u)}{\partial x} = 0$$

- **This equation does not involve cross sectional area of flow passage and hence applicable to only flow passage area is constant**

- **One-dimensional flow can also be assumed for non-uniform flow passage area if the flow velocity at each section is uniform**

- **For such a situation the continuity equation can be derived as follows:**

Continuity Equation for 1D flow



Let,
A = area at central plane
V = velocity at mid plane
ρ = density of fluid

Mass of fluid passing through mid plane

$$= \rho A V$$

Continuity Equation for 1D flow

Mass of fluid entering through plane NM per unit time

$$= \left[\rho AV - \frac{\partial}{\partial s}(\rho AV)\frac{\delta s}{2} \right]$$

Mass of fluid entering through plane N'M' per unit time

$$= \left[\rho AV + \frac{\partial}{\partial s}(\rho AV)\frac{\delta s}{2} \right]$$

Net mass of fluid that has remained in the fluid element per unit time

$$= -\frac{\partial}{\partial s}(\rho AV)\delta s$$

Mass of fluid element $= \rho A \delta s$

Rate of increase of mass of fluid element $= \dfrac{\partial}{\partial t}(\rho A)\delta s$

Continuity Equation for 1D flow

Thus

$$-\frac{\partial}{\partial s}(\rho AV)\delta s = \frac{\partial}{\partial t}(\rho A)\delta s$$

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho AV) = 0$$

For steady flow

$$\frac{\partial}{\partial s}(\rho AV) = 0 \quad \Rightarrow \quad \rho AV = \textbf{Constant}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \rho_3 A_3 V_2$$

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Continuity Equation for 1D flow

For incompressible fluid

$$AV = \textbf{constant} \Rightarrow A_1V_1 = A_2V_2 = A_3V_3 = \textbf{constant}$$

Further,

$$AV = q \ \textbf{is the dischage or volumetric flow}$$

- **This equation is applicable to steady one dimensional flow of an incompressible fluid**

- **Thus, for steady flow of an incompressible fluid discharge at any section is constant**

Continuity Equation for 1D flow

Continuity equation

$$A_1V_1 = A_2V_2 = A_3V_3 = \textbf{ constant}$$

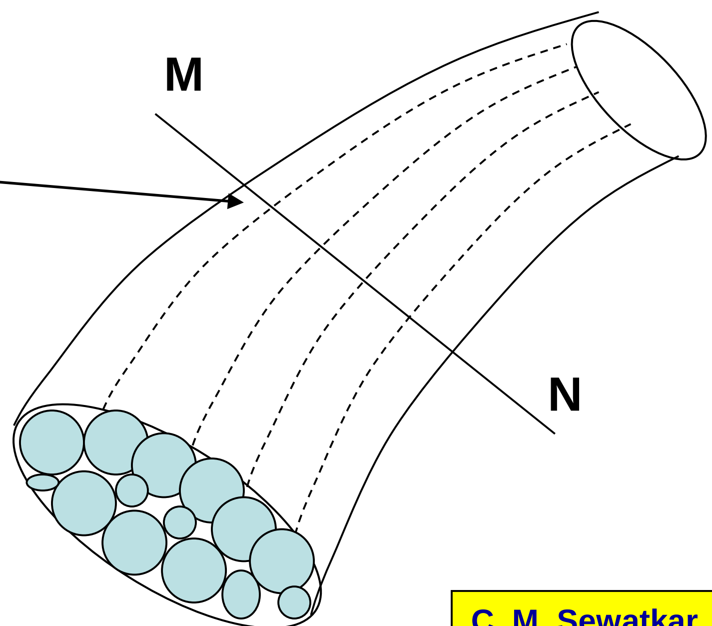
- **Derived for stream tube having small cross sectional areas A_1, A_2, A_3 etc having velocities V_1, V_2 and V_3; which are assumed to be uniform over a particular section**

- **However, the above equation can also be applied to flow passages of large areas, even if velocity is not uniform over a particular section i. e. it varies from point to point over a section**

dA_1, dA_2, dA_3

v_1, v_2 and v_3

dQ_1, dQ_2, dQ_3….



Continuity Equation for 1D flow

$$dQ_1 = v_1 dA_1,\ dQ_2 = v_2 dA_2,\ dQ_3 = v_3 dA_3 ....$$

$$Q = v_1 dA_1 + v_2 dA_2 + v_3 dA_3 ....$$

$$Q = \sum v dA$$

If V is the mean velocity at a particular section

$$Q = AV$$

Where $V = \frac{1}{A}\int v dA$ → **Mean velocity**

Tutorials

3.1 An incompressible fluid flows steadily through two pipes of diameter 15 cm and 20 cm which combine to discharge in a pipe of 30 cm diameter. If average velocities in 15 cm and 20 cm diameter pipes are 2 m/s and 3 m/s respectively, find the average velocity in 30 cm diameter pipe

3.2. Determine which of the following pairs of velocity components satisfy continuity equation for two dimensional flow of an incompressible fluid

a) $u = Cx\ ;\ v = -Cy$

b) $u = 3x - y;\ v = 2x + 3y$

c) $u = x + y;\ v = x^2 - y$

d) $u = A\sin xy;\ v = -A\sin xy$

e) $u = 2x^2 + 3y^2;\ v = -3xy$

Tutorials

3.3 Determine unknown velocity component so that those satisfy the continuity equation. Make suitable assumptions.

a) $u = 2x^2 \; ; v = xyz; \; w = ?$

b) $u = 2x^2 + 2xy; \; w = z^3 - 4xz - 2yz; \; v = ?$

Tutorials

Consider the cube with 1 m edges parallel to the coordinate axes located in the first quadrant with one corner at the origin. By using the velocity distribution of .

$$\mathbf{V} = (5x)\mathbf{i} + (5y)\mathbf{j} - (10z)\mathbf{k}$$

Find the flow through each face and show that no mass is accumulated within the cube if fluid is of constant density.

Acceleration of Fluid Particle

Rate of change of velocity

$$a_x = \lim_{dt \to 0} \frac{du}{dt};\ a_y = \lim_{dt \to 0} \frac{dv}{dt};\ a_z = \lim_{dt \to 0} \frac{dw}{dt};$$

We known $u = f(x, y, z, t)$

Total derivative of u using partial derivative is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

$$\lim_{dt \to 0} \frac{dx}{dt} = u\ ,\ \lim_{dt \to 0} \frac{dy}{dt} = v\ ,\ \lim_{dt \to 0} \frac{dz}{dt} = w$$

Acceleration of Fluid Particle

Thus

$$\lim_{dt \to 0} \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

In vector notation the acceleration of a fluid particle is written as:

$$a = V.\nabla V$$

Acceleration of Fluid Particle

$$\frac{\partial u}{\partial t} \qquad \frac{\partial v}{\partial t} \qquad \frac{\partial w}{\partial t}$$

Rate of change of velocity with respect to time at a particular point

Local or temporal acceleration

$$\left.\begin{matrix} u\dfrac{\partial u}{\partial x};\ v\dfrac{\partial u}{\partial y};\ w\dfrac{\partial u}{\partial z}; \\ u\dfrac{\partial v}{\partial x};\ v\dfrac{\partial v}{\partial y};\ w\dfrac{\partial v}{\partial z}; \\ u\dfrac{\partial w}{\partial x};\ v\dfrac{\partial w}{\partial y};\ w\dfrac{\partial w}{\partial z}; \end{matrix}\right\}$$

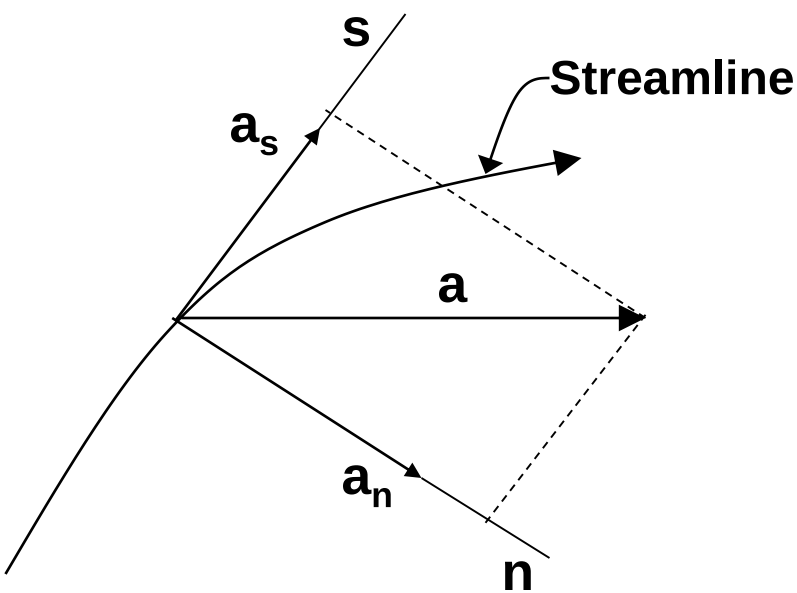
Increase in velocity due to change in position of particle

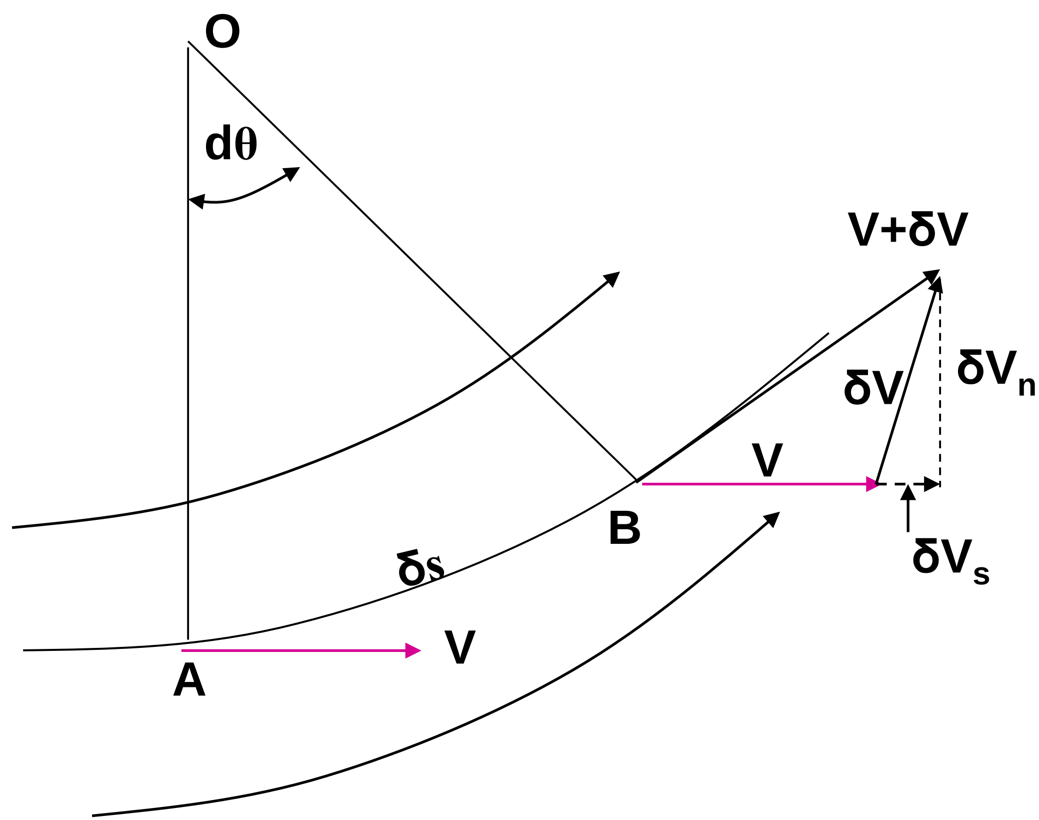
Convective of spatial acceleration

Tangential and Normal Accelerations

- Like velocity acceleration is also a vector

- However, acceleration has no fixed orientation with streamline





Let V_s and V_n be the components of velocity along tangential and normal directions

Tangential and Normal Accelerations

$$V_s = f_1(s,n,t) \quad \textbf{and} \quad V_n = f_2(s,n,t)$$

The accelerations in tangential and normal directions may be expressed as:

$$a_s = \lim_{dt \to 0} \frac{dV_s}{dt} \quad \textbf{and} \quad a_n = \lim_{dt \to 0} \frac{dV_n}{dt}$$

The tangential component is due to change in the magnitude of velocity along the streamline

The normal component is due to change in the direction of velocity vector

Tangential and Normal Accelerations

$$\frac{dV_s}{dt} = \frac{\partial V_s}{\partial s}\frac{ds}{dt} + \frac{\partial V_s}{\partial n}\frac{dn}{dt} + \frac{\partial V_s}{\partial t}\frac{dt}{dt}$$

and

$$\frac{dV_n}{dt} = \frac{\partial V_n}{\partial s}\frac{ds}{dt} + \frac{\partial V_n}{\partial n}\frac{dn}{dt} + \frac{\partial V_n}{\partial t}\frac{dt}{dt}$$

We know $\lim_{dt\to 0}\frac{ds}{dt} = V_s$ **and** $\lim_{dt\to 0}\frac{dn}{dt} = V_n$

$$\lim_{dt\to 0}\frac{dV_s}{dt} = a_s = V_s\frac{\partial V_s}{\partial s} + V_n\frac{\partial V_s}{\partial n} + \frac{\partial V_s}{\partial t}$$

$$\lim_{dt\to 0}\frac{dV_n}{dt} = a_n = V_s\frac{\partial V_n}{\partial s} + V_n\frac{\partial V_n}{\partial n} + \frac{\partial V_n}{\partial t}$$

Tangential and Normal Accelerations

For a given streamline $V_n = 0$

$$a_s = V_s \frac{\partial V_s}{\partial s} + \frac{\partial V_s}{\partial t}$$

$$a_n = V_s \frac{\partial V_n}{\partial s} + \frac{\partial V_n}{\partial t}$$

Note that though $V_n = 0$ $\frac{\partial V_n}{\partial s}$ **is not equal to zero**

V_n is zero at any point on the streamline but at any other point on the streamline the component of the velocity in the direction parallel to the of V_n need not be always zero

Tangential and Normal Accelerations

Further

$$d\theta = \frac{\delta s}{r} = \frac{\partial V_n}{V}$$

$$\textbf{or} \quad \frac{\partial V_n}{\delta s} = \frac{\partial V_n}{\partial s} = \frac{V}{r} = \frac{V_s}{r} \quad \textbf{(since } V = V_s\textbf{)}$$

$$a_n = \frac{V_s^2}{r} + \frac{\partial V_n}{\partial t}$$

Tangential and Normal Accelerations

$$\frac{\partial V_s}{\delta t}$$ → **Local tangential acceleration**

$$\frac{\partial V_n}{\delta t}$$ → **Local normal acceleration**

} **Zero for steady flow**

$$V_s \frac{\partial V_s}{\delta s}$$ → **Convective tangential acceleration**

$$V_s \frac{\partial V_n}{\delta s} = \frac{V_s^2}{r}$$ → **Convective normal acceleration**

For steady flow →

$$a_s = V_s \frac{\partial V_s}{\partial s}$$

$$a_n = \frac{V_s^2}{r}$$

Tangential and Normal Accelerations

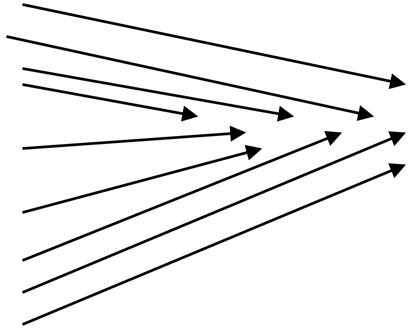
Straight streamline, $r = \infty$ **Hence no normal acceleration**

Convective normal acceleration is developed only of the flow is along curved path so that streamline are curved

Straight and parallel streamlines

- **Convective tangential acceleration is zero**
- **Thus, No acceleration**

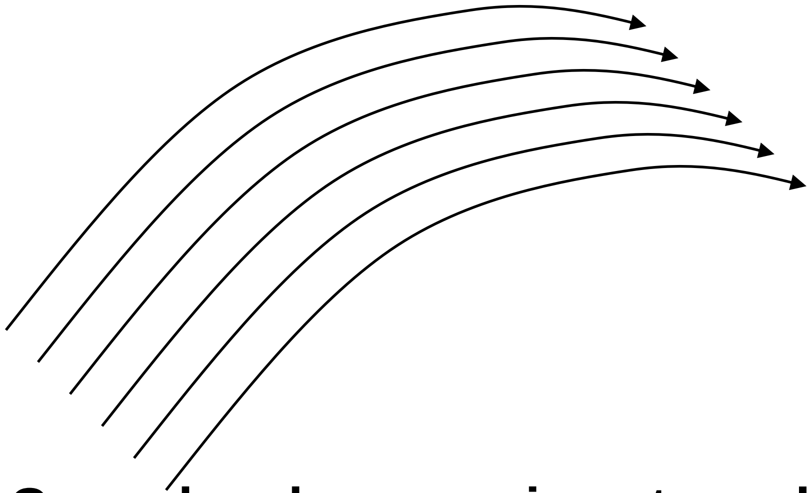
Straight and converging streamlines



- **Convective tangential acceleration is non-zero**

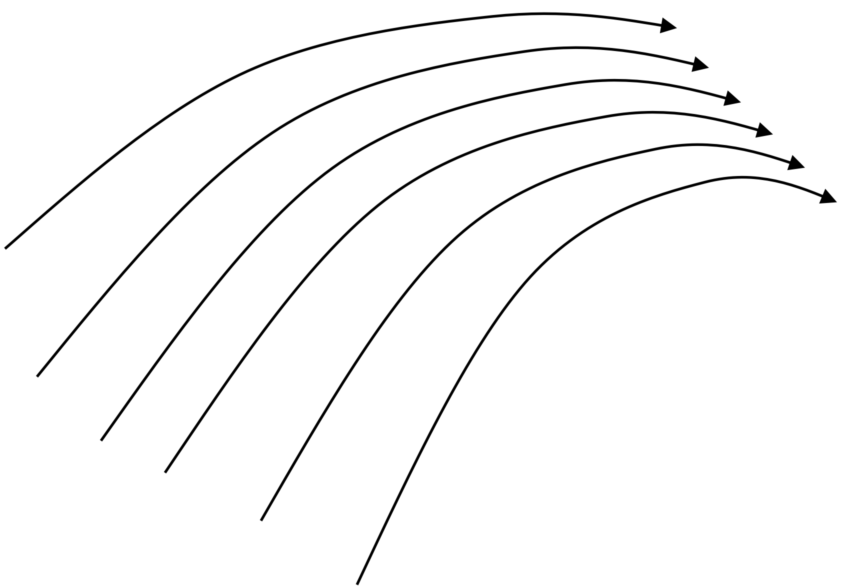
Tangential and Normal Accelerations

Curved equidistance streamlines



Convective tangential acceleration is zero and only normal convective acceleration will be there

Curved and converging streamlines

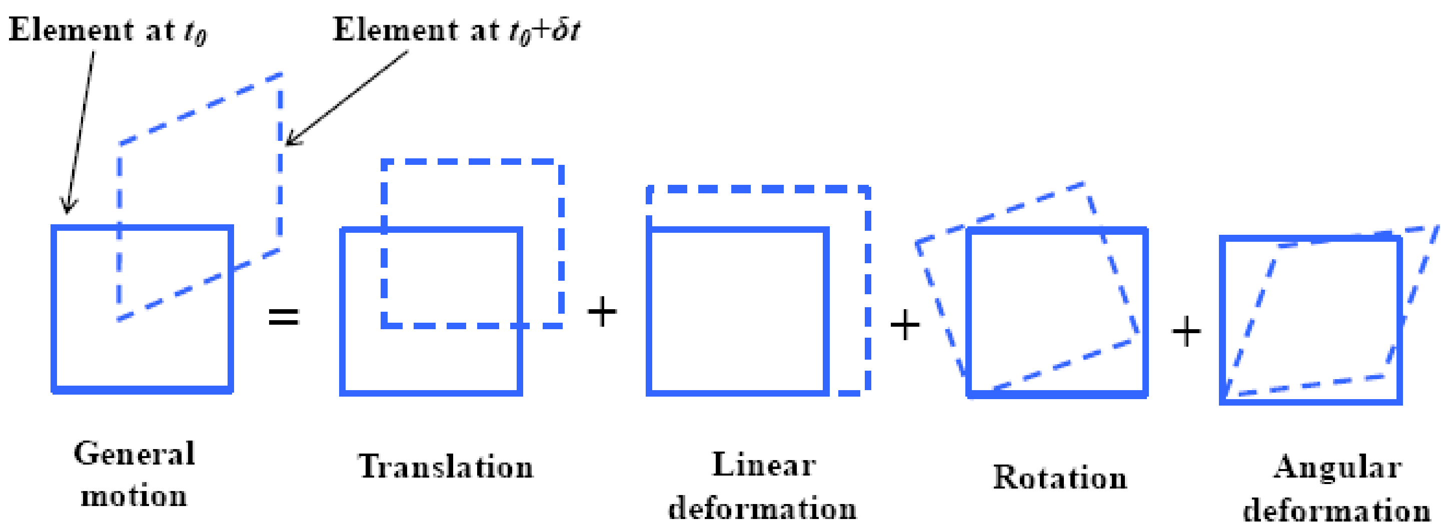


Convective tangential acceleration Convective normal acceleration will be there

Rotational and Irrotational Flow

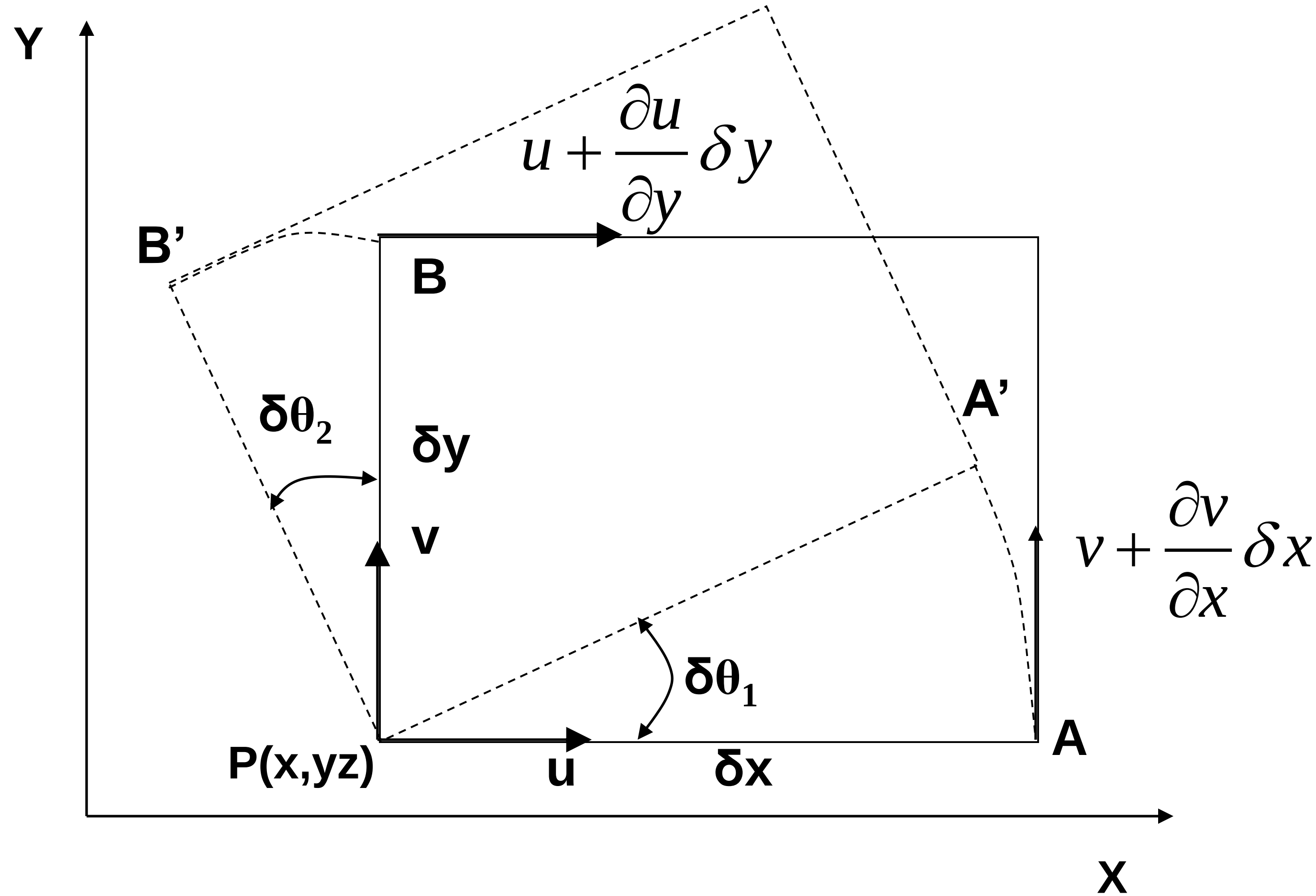
Fundamental Motions of a Fluid Particles

- ❑Linear Translation or Pure Translation
- ❑Linear Deformation
- ❑Angular Deformation
- ❑Rotation



Rotational and Irrotational Flow

The rotation of fluid particle may be defined in terms of Component of Rotation about three mutually perpendicular axes.



Rotational and Irrotational Flow

We can write

$$\omega_{PA} = \lim_{\delta t \to 0} \frac{\delta \theta_1}{\delta t}$$

θ = s/r

$$\omega_{PA} = \lim_{\delta t \to 0} \frac{\left[\left(v + \frac{\partial v}{\partial x}\delta x\right) - v\right]\delta t}{\delta x \ \delta t}$$

Thus,

$$\omega_{PA} = \frac{\partial v}{\partial x}$$

Rotational and Irrotational Flow

Also

$$\omega_{PB} = \lim_{\delta t \to 0} \frac{\delta \theta_2}{\delta t}$$

θ = s/r

$$\omega_{PB} = \lim_{\delta t \to 0} \frac{-\left[\left(u + \frac{\partial u}{\partial y}\delta y\right) - u\right]\delta t}{\delta y \ \delta t}$$

Thus,

$$\omega_{PB} = -\frac{\partial u}{\partial y}$$

Rotational and Irrotational Flow

The component of rotation – the average angular velocity of two infinitesimally linear elements in the particle that are perpendicular to each other and to the axis of rotation (In this case Z axis)

Thus, Component of Rotation about Z axis

$$\omega_z = \frac{1}{2}(\omega_{PA} + \omega_{PB})$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Rotational and Irrotational Flow

If the component of rotation for all the axes is zero the flow is said to be irrotational else it is rotational

Thus, for flow to be irrotational

$$\omega_x = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$

$$\omega_y = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

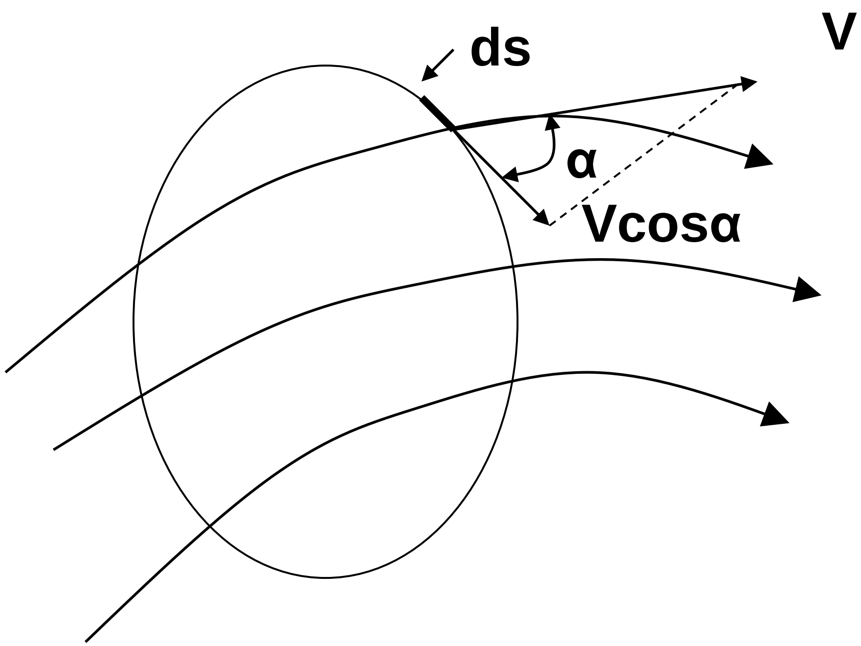
$$\omega_z = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

The rotation of fluid is always associated with shear stress

Circulation and Vorticity

The flow along a closed curve is called circulation i. e. flow in eddied and vortices

Mathematically, circulation is the line integral, taken around a closed curve, of the tangential component of velocity vector

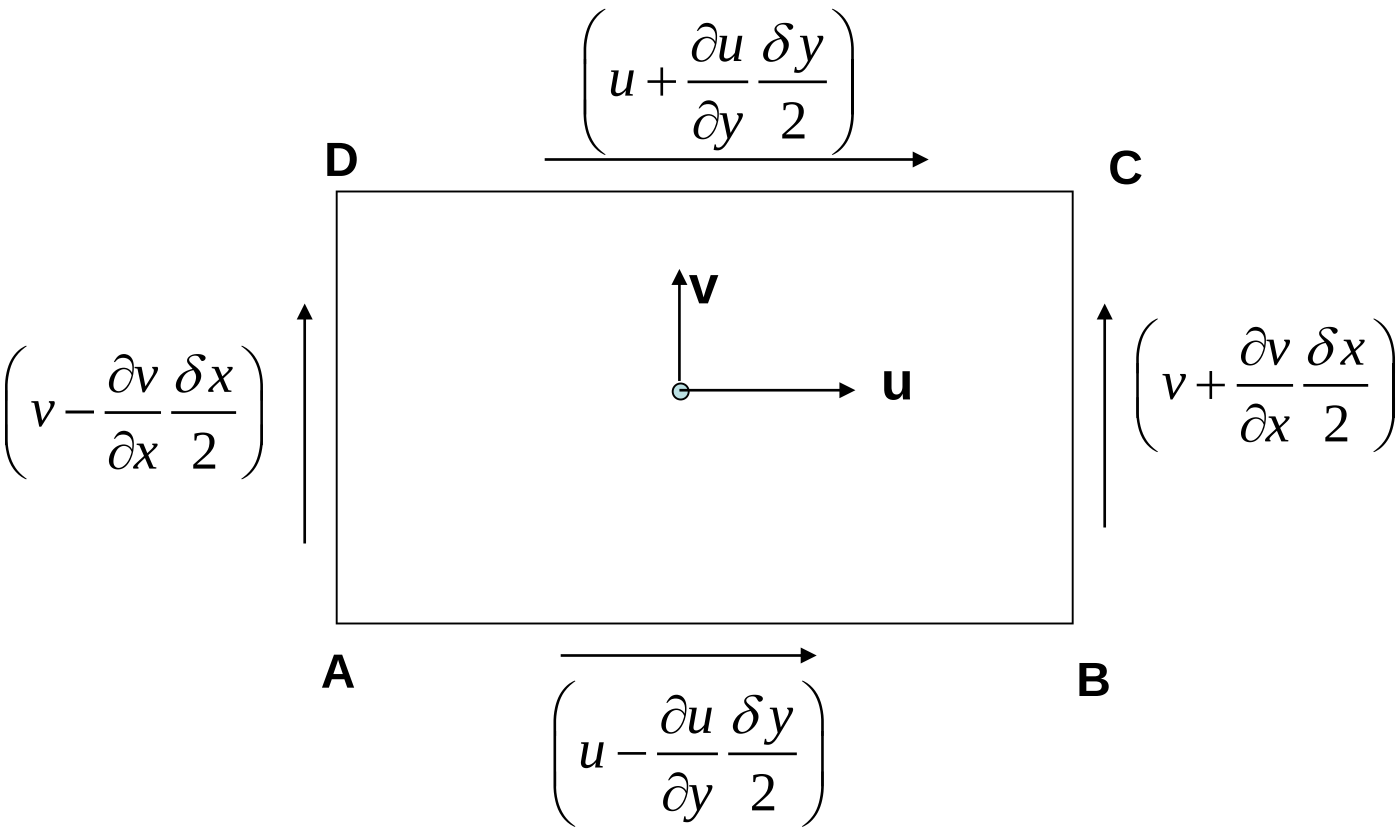


$$\lambda = \int_C V \cos\alpha \; ds$$

$$\lambda = \int_C (udx + vdy + wdz)$$

Circulation and Vorticity

Circulation around an elementary rectangle



Circulation along AB = $\left(u-\frac{\partial u}{\partial y}\frac{\delta y}{2}\right)\delta x$

Circulation and Vorticity

Circulation along BC = $\left(v + \frac{\partial v}{\partial x}\frac{\delta x}{2}\right)\delta y$

Circulation along CD = $-\left(u + \frac{\partial u}{\partial y}\frac{\delta y}{2}\right)\delta x$

Circulation along DA = $-\left(v - \frac{\partial v}{\partial x}\frac{\delta x}{2}\right)\delta y$

What ever may be the shape of the curve the circulation must be equal to the sum of the circulation around the elementary surfaces of which it consists, provided the boundary of the curve is wholly in the fluid

Circulation and Vorticity

Thus,

$$\lambda = \lambda_{AB} + \lambda_{BC} + \lambda_{CD} + \lambda_{DA}$$

$$\lambda = \left(u - \frac{\partial u}{\partial y}\frac{\delta y}{2}\right)\delta x + \left(v + \frac{\partial v}{\partial x}\frac{\delta x}{2}\right)\delta y - \left(u + \frac{\partial u}{\partial y}\frac{\delta y}{2}\right)\delta x - \left(v - \frac{\partial v}{\partial x}\frac{\delta x}{2}\right)\delta y$$

$$\lambda = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x \delta y$$

The vorticity at any is defined as the ratio of the circulation around an infinitesimal closed curve at any point to the area of the curve

Circulation and Vorticity

Thus, vorticity is given as

$$\xi = \omega = \frac{Circulation}{Area}$$

$$\xi = \omega = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\xi = 2\omega_z$$

Some facts about Vorticity

Vorticity is the vector quantity whose direction is perpendicular to the plane of the small curve round which the circulation is measured

Thus,

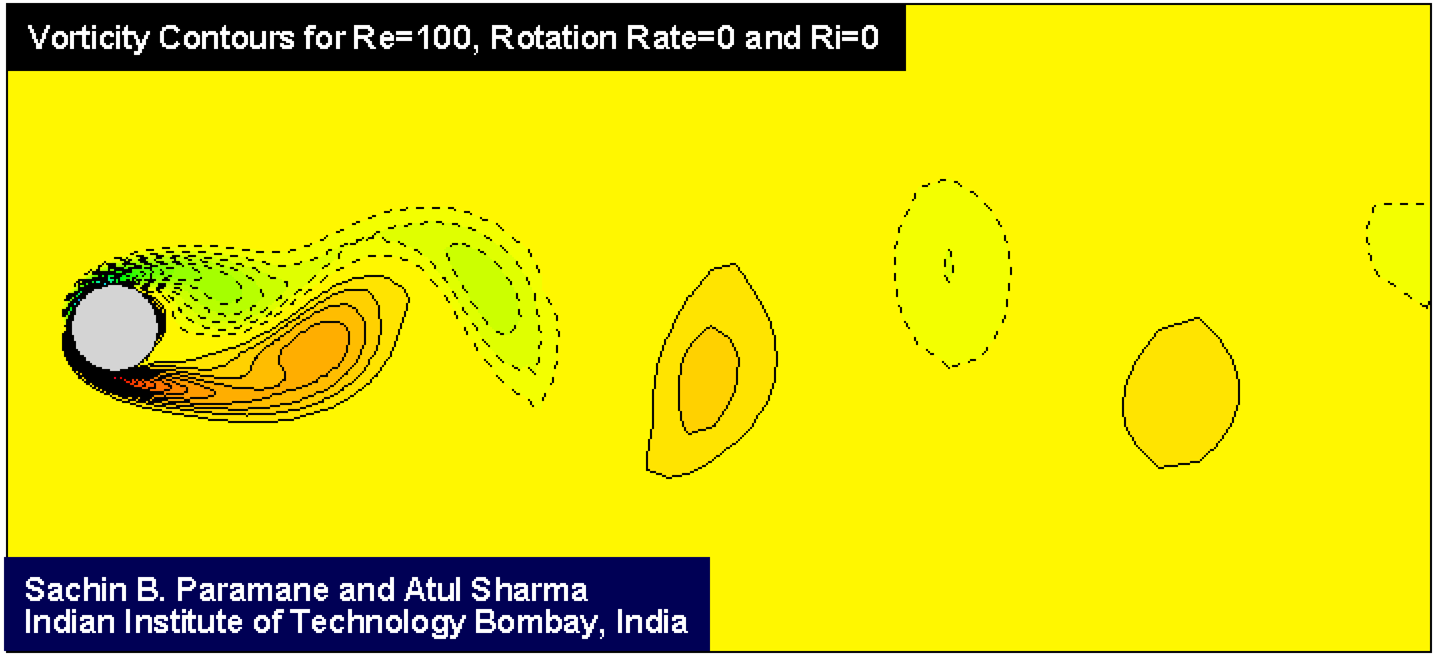
$$\xi_x = 2\omega_x \quad \xi_y = 2\omega_y \quad \xi_z = 2\omega_z$$

If vorticity is zero at all points in a region then the flow in that region is said to be irrotational.

In vector notation vorticity may be written as

$$\xi = \nabla \times V \Rightarrow \xi = curl\ V$$

Simple Bluff Body Flow Problem



Velocity Potential

The velocity potential Φ (phi) is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the velocity in that direction

Mathematically, if Φ = f(x, y, z, t)

$$u = -\frac{\partial \phi}{\partial x}; \quad v = -\frac{\partial \phi}{\partial y} \quad \textbf{and} \quad w = -\frac{\partial \phi}{\partial w}$$

The negative sign indicates that Φ decreases with an increase in values of x, y, z, Thus, flow is always in the direction of decreasing Φ.

Velocity Potential

For steady flow of incompressible fluid continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right)=0$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

Laplace Equation

$$\nabla^2 \phi = 0$$

Any function Φ which satisfies the Laplace equation is the possible case of flow

Velocity Potential

Further, for a rotational flow components of rotation are given as:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \Rightarrow \omega_x = \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \Rightarrow \omega_y = \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \Rightarrow \omega_z = \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

Velocity Potential

If Φ is a continuous function

$$\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y} \qquad \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z} \qquad \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

- **Thus, any function that satisfies Laplace equation is possible case of irrotational flow since continuity is satisfied**

- **Velocity potential exists only for irrotational flows of fluids.**

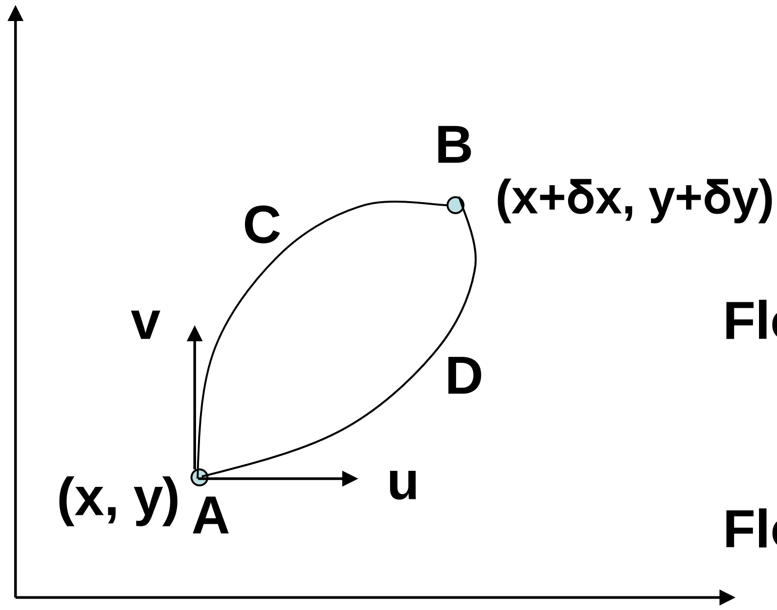
- **Hence irrotational flow is often called potential flow.**

Stream Function

The stream function Ψ (psi) is defined as a scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles (in the counter clockwise direction) to this direction

Mathematically, if Ψ = f(x, y, t)

$$\frac{\partial \psi}{\partial x} = v; \qquad \frac{\partial \psi}{\partial y} = -u$$



Flow across the curve ACB in x-direction

$$= -u\delta y$$

Flow across the curve ACB in y-direction

$$= v\delta x$$

Stream Function

If dΨ is assumed to represent the total flow across ACB

$$d\psi = -u\delta y + v\delta x$$ ——— **Equation A**

If fluid is homogeneous and incompressible the flow across ADB or any other curve must be same as that across ACB

For steady flow the fundamental definition of stream function suggests that Ψ = f(x, y)

$$d\psi = \frac{\partial \psi}{\partial x}\delta x + \frac{\partial \psi}{\partial y}\delta y$$ ——— **Equation B**

Stream Function

Compare A and B

$$\frac{\partial \psi}{\partial x} = v \quad \textbf{and} \quad \frac{\partial \psi}{\partial y} = -u$$

Thus, the assumption that dΨ is the flow across two points is valid and stream function can be used for determination of flow between two points if the stream functions at these points is known.

Compare the above equations with the equations for velocity potential

$$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y} \quad \textbf{and} \quad -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}$$

Stream Function

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \textbf{and} \quad -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

— **Cauchy-Riemann Equations**

Further, we know that

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

— **Poisson's equation**

For irrotational flow

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Laplace equation for Ψ

Stream Function

Further, the continuity equation for steady flow of incompressible fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0$$

$$\frac{\partial^2 \psi}{\partial x \partial y}=\frac{\partial^2 \psi}{\partial y \partial x}$$

Thus if Ψ is the continuous function and its second derivative exists it can be a possible case of flow since it satisfies the continuity equation

Streamlines, Equipotential Lines and Flow Net

Property of stream function – The difference of its values at two pints gives the flow across any line joining the two points.

Thus, if two points are on same streamline and since there is no flow across the streamline, the values of stream function at these two points will be same i. e. $\Psi_1 = \Psi 2$

Thus, a streamline can be represented by Ψ = constant

Similarly, Φ = constant represents a curve for which velocity potential is constant; such a curve is called equipotential line.

Streamlines, Equipotential Lines and Flow Net

Consider the slope of streamline and equipotential line at intersection in a flow domain

For Φ = constant

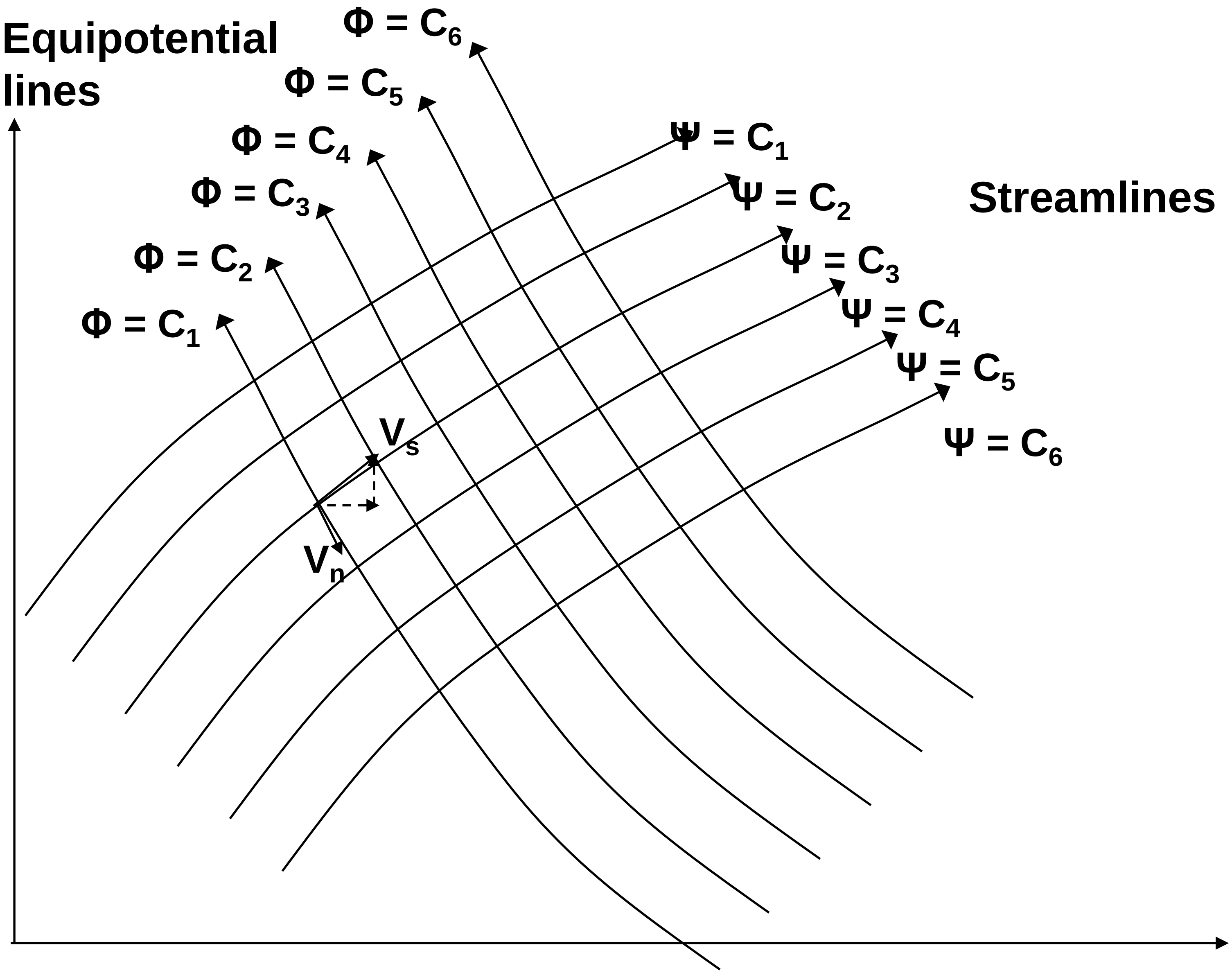
$$\textbf{slope} = \frac{\partial y}{\partial x} = \frac{\left(\dfrac{\partial \phi}{\partial x}\right)}{\left(\dfrac{\partial \phi}{\partial y}\right)} = \frac{-u}{-v} = \frac{u}{v}$$

For Ψ = constant

$$\textbf{slope} = \frac{\partial y}{\partial x} = \frac{\left(\dfrac{\partial \psi}{\partial x}\right)}{\left(\dfrac{\partial \psi}{\partial y}\right)} = \frac{v}{-u} = -\frac{v}{u}$$

The product of slopes of two lines is -1, which means streamline and equipotential line intersect orthogonally

Streamlines, Equipotential Lines and Flow Net



Streamlines, Equipotential Lines and Flow Net

We know

$$-\frac{\partial \phi}{\partial n} = V_n \quad \Rightarrow \quad V_n = 0$$

Φ is constant along n-direction

$$-\frac{\partial \phi}{\partial s} = V_s \quad \Rightarrow$$

The flow is along streamline as Φ is not constant along s-direction

Similarly

$$-\frac{\partial \psi}{\partial s} = V_n = 0$$

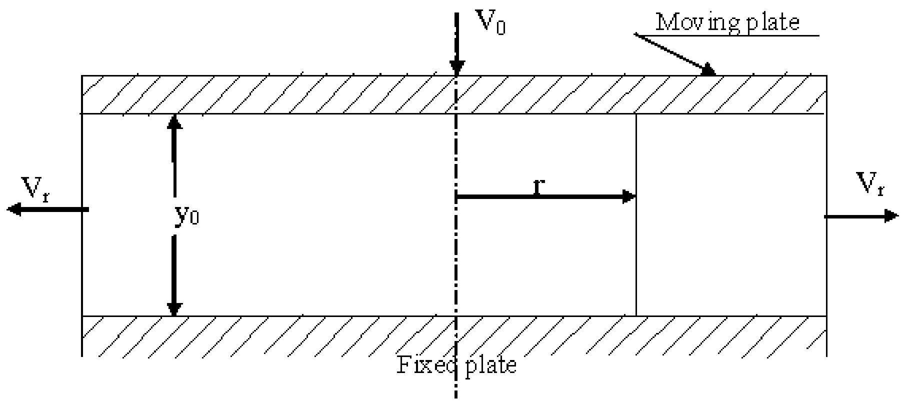
$$\frac{\partial \psi}{\partial n} = -V_s$$

$\Rightarrow$ **No flow in the direction normal to streamline but the flow is always along streamline**

Assignment – Prepare a descriptive note on the methods of drawing the flow net and attach it in your lab practice journal.

Tutorials

Two large circular plates are kept at a distance y_0 apart and contain an incompressible fluid in between. If the bottom plate is fixed and top plate is moved downward at a constant velocity of V_0, estimate the velocity at which the fluid moves at a radial distance of r.



Tutorials

A 2.0 m long diffuser 20 cm in diameter at the upstream end has 80 cm diameter at the downstream end. At a certain instant the discharge through the diffuser is observed to be 200 liters/s of water and is found to increase uniformly at a rate of 50 liters/s per second. Estimate the local, convective and total acceleration at a section 1.5 m from the upstream end.

Tutorials

For two-dimensional incompressible flow, show that the flow rate per unit width between two streamlines is equal to the difference between the values of stream function corresponding to these streamlines.

Tutorials

The velocity profile as a function of radius is given as;

$$u = u_m \left[1 - (r/R)\right]^n$$

where R is the radius of the pipe and u_m is the maximum velocity. Calculate the average or mean velocity for n = 1/5 and n = ½ in terms of u_m